CHAPTER 12

CHAPTER 12 Circles

Chapter Outline

Finally, we dive into a different shape, circles. First, we will define all the parts of circles and then explore the properties of tangent lines, arcs, inscribed angles, and chords. Next, we will learn about the properties of angles within circles that are formed by chords, tangents and secants. Lastly, we will place circles in the coordinate plane, find the equations of, and graph circles.

12.1 Parts of Circles & Tangent Lines

Learning Objectives

- Define circle, center, radius, diameter, chord, tangent, and secant of a circle.
- Explore the properties of tangent lines and circles.

Review Queue

a. Find the equation of the line with $m = -2$ and passes through (4, -5).
b. Find the equation of the line that passes though (6, 2) and (-3, -1).
c. Find the equation of the line *perpendicular* to the line in #2 and passes through (-8, 11).

Know What? The clock to the right is an ancient astronomical clock in Prague. It has a large background circle that tells the local time and the "ancient time" and then the smaller circle rotates around on the orange line to show the current astrological sign. The yellow point is the center of the larger clock. How does the orange line relate to the small and larger circle? How does the hand with the moon on it (black hand with the circle) relate to both circles? Are the circles concentric or tangent?

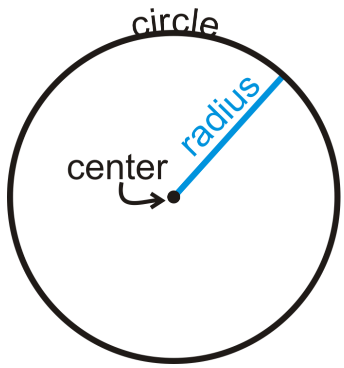
For more information on this clock, see: http://en.wikipedia.org/wiki/Prague_Astronomical_Clock

Defining Terms

Circle: The set of all points that are the same distance away from a specific point, called the **center.**

Radius: The distance from the center to the circle.

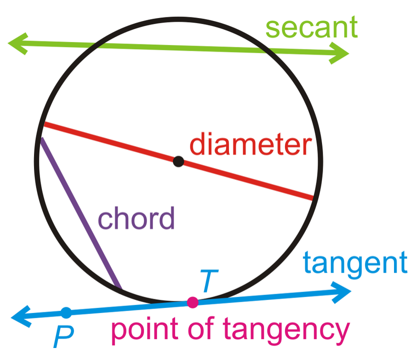
The center is typically labeled with a capital letter because it is a point. If the center is A, we would call this circle, "circle A," and labeled $\odot A$. Radii (the plural of radius) are line segments. There are infinitely many radii in any circle.



Chord: A line segment whose endpoints are on a circle.

Diameter: A chord that passes through the center of the circle.

Secant: A line that intersects a circle in two points.

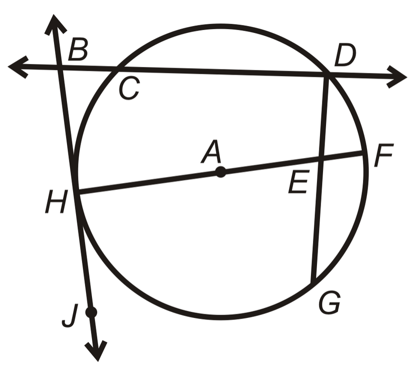


Tangent: A line that intersects a circle in exactly one point.

Point of Tangency: The point where the tangent line touches the circle.

Notice that the tangent ray $\overrightarrow{TP}$ and tangent segment $\overline{TP}$ are also called tangents. The length of a diameter is two times the length of a radius.

Example 1: Identify the parts of $\odot A$ that best fit each description.



a) A radius

b) A chord

c) A tangent line

d) The point of tangency

e) A diameter

f) A secant

Solution:

a) $\overline{HA}$ or $\overline{AF}$

b) $\overline{CD}, \overline{HF}$, or $\overline{DG}$

c) $\overleftrightarrow{BJ}$

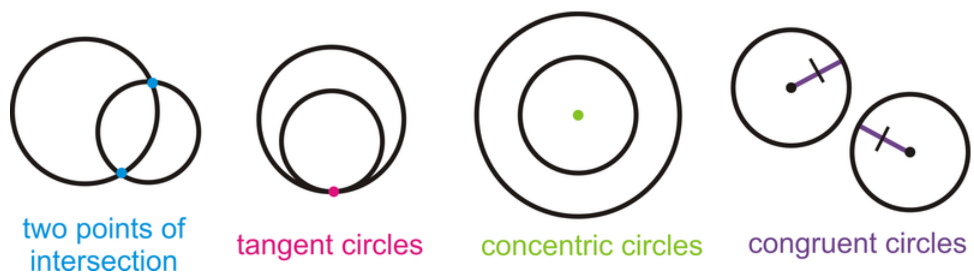
d) *Point H*

e) $\overline{HF}$

f) $\overleftrightarrow{BD}$

Coplanar Circles

Two circles can intersect in two points, one point, or no points. If two circles intersect in one point, they are called *tangent circles.*

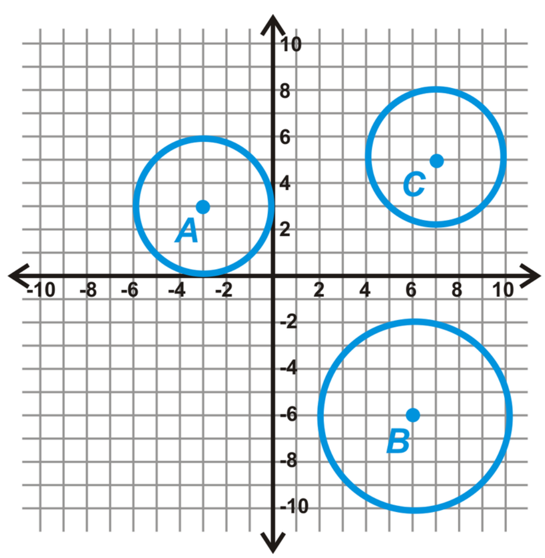


Congruent Circles: Two circles with the same radius, but different centers.

Concentric Circles: When two circles have the same center, but different radii.

If two circles have different radii, they are similar. *All circles are similar.*

Example 2: Determine if any of the following circles are congruent.



Solution: From each center, count the units to the circle. It is easiest to count vertically or horizontally. Doing this, we have:

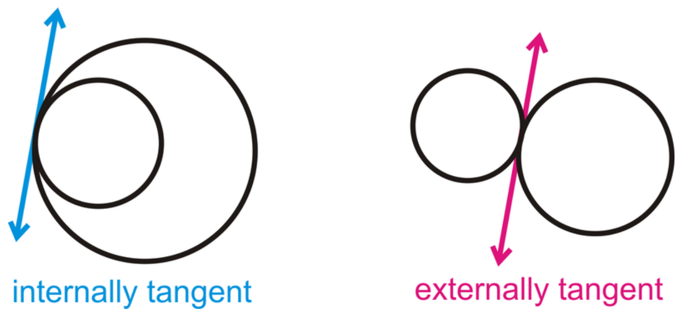
Radius of $\odot A = 3\ units$
Radius of $\odot B = 4\ units$
Radius of $\odot C = 3\ units$

From these measurements, we see that $\odot A \cong \odot C$.

Notice that two circles are congruent, just like two triangles or quadrilaterals. Only the *lengths* of the radii are equal.

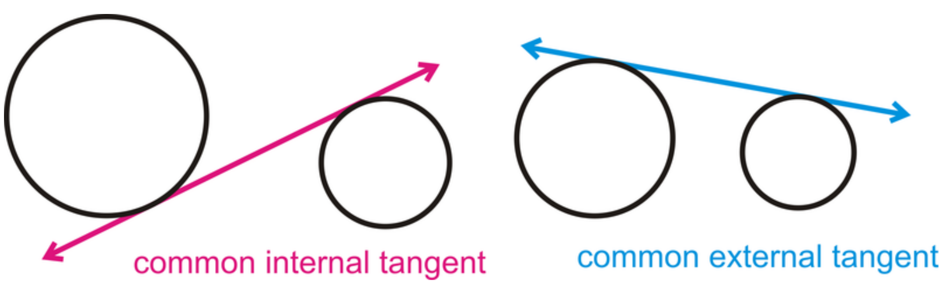
Tangent Lines

We just learned that two circles can be tangent to each other. Two triangles can be tangent in two different ways, either *internally* tangent or *externally* tangent.



If the circles are not tangent, they can share a tangent line, called a *common* tangent. Common tangents can be internally tangent and externally tangent too.

Notice that the common internal tangent passes through the space between the two circles. Common external tangents stay on the top or bottom of both circles.



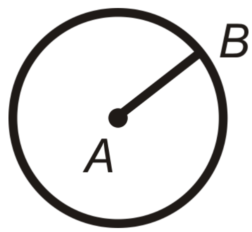
Tangents and Radii

The tangent line and the radius drawn to the point of tangency have a unique relationship. Let's investigate it here.

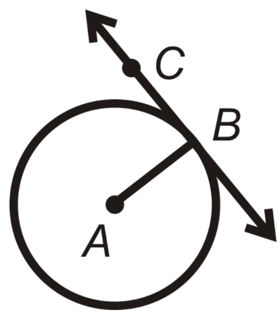
Investigation 9-1: Tangent Line and Radius Property

Tools needed: compass, ruler, pencil, paper, protractor

a. Using your compass, draw a circle. Locate the center and draw a radius. Label the radius $\overline{AB}$, with A as the center.



b. Draw a tangent line, $\overleftrightarrow{BC}$, where B is the point of tangency. To draw a tangent line, take your ruler and line it up with point B. Make sure that B is the only point on the circle that the line passes through.

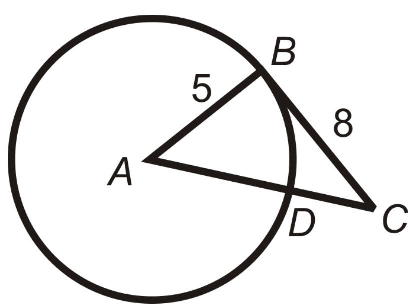


c. Using your protractor, find $m\angle ABC$.

Tangent to a Circle Theorem: A line is tangent to a circle if and only if the line is perpendicular to the radius drawn to the point of tangency.

To prove this theorem, the easiest way to do so is indirectly (proof by contradiction). Also, notice that this theorem uses the words "if and only if," making it a biconditional statement. Therefore, the converse of this theorem is also true.

Example 3: In $\odot A, \overline{CB}$ is tangent at point B. Find AC. Reduce any radicals.



Solution: Because $\overline{CB}$ is tangent, $\overline{AB} \perp \overline{CB}$, making $\triangle ABC$ a right triangle. We can use the Pythagorean Theorem to find AC.

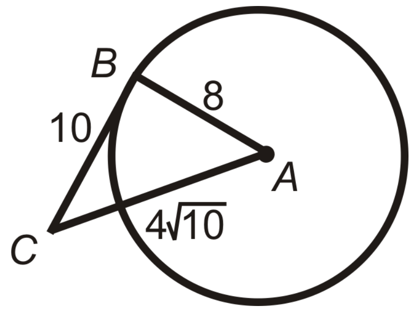
$$\begin{aligned}
5^2 + 8^2 &= AC^2 \\
25 + 64 &= AC^2 \\
89 &= AC^2 \\
AC &= \sqrt{89}
\end{aligned}$$

Example 4: Find DC, in $\odot A$. Round your answer to the nearest hundredth.

Solution:

$$DC = AC - AD$$
$$DC = \sqrt{89} - 5 \approx 4.43$$

Example 5: Determine if the triangle below is a right triangle. Explain why or why not.

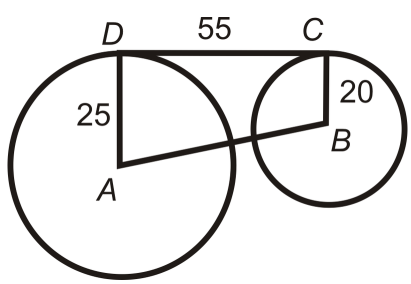


Solution: To determine if the triangle is a right triangle, use the Pythagorean Theorem. $4\sqrt{10}$ is the longest length, so we will set it equal to c in the formula.

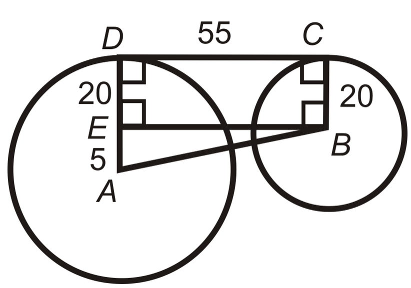
$$8^2 + 10^2 \; ? \; \left(4\sqrt{10}\right)^2$$
$$64 + 100 \neq 160$$

$\triangle ABC$ is not a right triangle. And, from the converse of the Tangent to a Circle Theorem, $\overline{CB}$ is not tangent to $\odot A$.

Example 6: Find the distance between the centers of the two circles. Reduce all radicals.



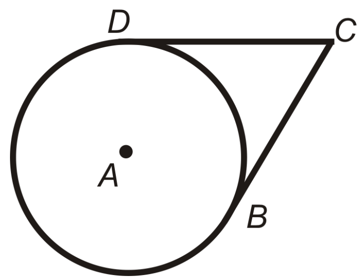
Solution: The distance between the two circles is AB. They are not tangent, however, $\overline{AD} \perp \overline{DC}$ and $\overline{DC} \perp \overline{CB}$. Let's add $\overline{BE}$, such that $EDCB$ is a rectangle. Then, use the Pythagorean Theorem to find AB.



$$\begin{aligned} 5^2 + 55^2 &= AC^2 \\ 25 + 3025 &= AC^2 \\ 3050 &= AC^2 \\ AC &= \sqrt{3050} = 5\sqrt{122} \end{aligned}$$

Tangent Segments

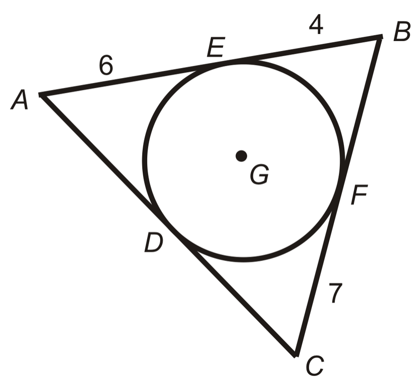
Let's look at two tangent segments, drawn from the same external point. If we were to measure these two segments, we would find that they are equal.



Theorem 10-2: If two tangent segments are drawn from the same external point, then the segments are equal.

The proof of Theorem 10-2 is in the review exercises.

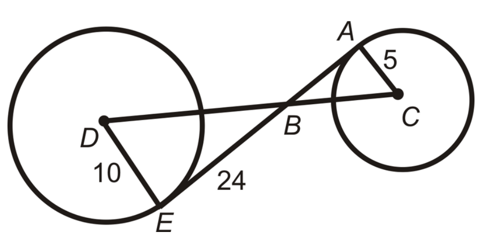
Example 7: Find the perimeter of $\triangle ABC$.



Solution: $AE = AD, EB = BF$, and $CF = CD$. Therefore, the perimeter of $\triangle ABC = 6+6+4+4+7+7 = 34$.

We say that $\odot G$ is ***inscribed*** in $\triangle ABC$. A circle is inscribed in a polygon, if every side of the polygon is tangent to the circle.

Example 8: If D and A are the centers and AE is tangent to both circles, find DC.



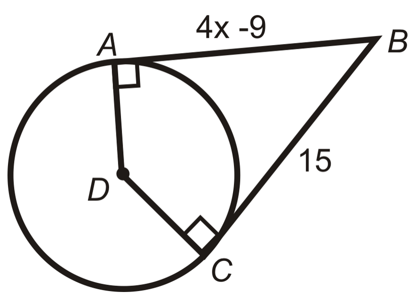
Solution: Because AE is tangent to both circles, it is perpendicular to both radii and $\triangle ABC$ and $\triangle DBE$ are similar. To find DB, use the Pythagorean Theorem.

$$\begin{aligned}10^2+24^2 &= DB^2\\100+576 &= 676\\DB &= \sqrt{676}=26\end{aligned}$$

To find BC, use similar triangles.

$$\frac{5}{10}=\frac{BC}{26}\longrightarrow BC=13$$
$$DC=AB+BC=26+13=39$$

Example 9: ***Algebra Connection*** Find the value of x.



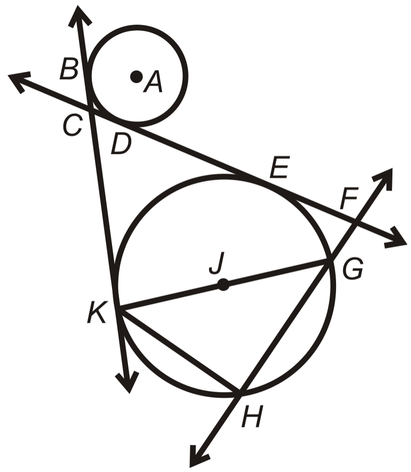
Solution: Because $\overline{AB}\perp\overline{AD}$ and $\overline{DC}\perp\overline{CB}, \overline{AB}$ and $\overline{CB}$ are tangent to the circle and also congruent. Set $AB=CB$ and solve for x.

$$\begin{aligned}4x-9 &= 15\\4x &= 24\\x &= 6\end{aligned}$$

Know What? Revisited Refer to the photograph in the "Know What?" section at the beginning of this chapter. The orange line (which is normally black, but outlined for the purpose of this exercise) is a diameter of the smaller circle. Since this line passes through the center of the larger circle (yellow point, also outlined), it is part of one of its diameters. The "moon" hand is a diameter of the larger circle, but a secant of the smaller circle. The circles are not concentric because they do not have the same center and are not tangent because the sides of the circles do not touch.

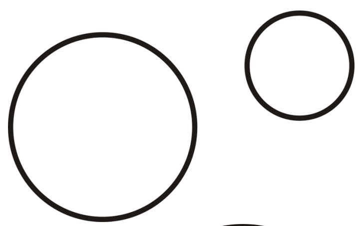
Review Questions

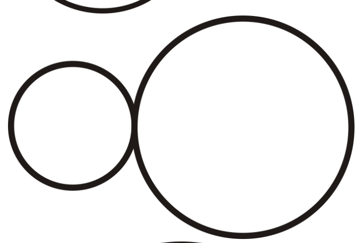
Determine which term best describes each of the following parts of $\odot P$.

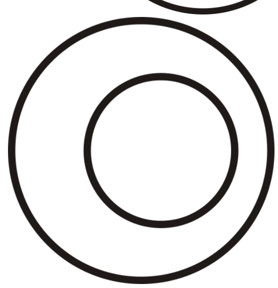


1. $\overline{KG}$
2. $\overleftrightarrow{FH}$
3. $\overline{KH}$
4. E
5. $\overleftrightarrow{BK}$
6. $\overleftrightarrow{CF}$
7. A
8. $\overline{JG}$
9. What is the longest chord in any circle?

Copy each pair of circles. Draw in all common tangents.

10. 

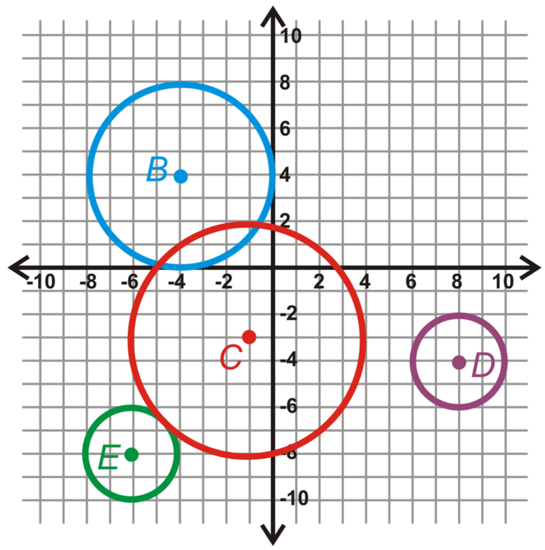
11. 

12. 

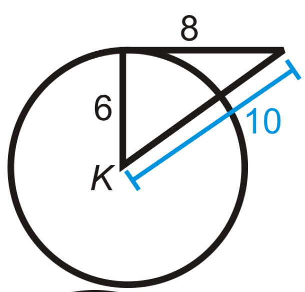
Coordinate Geometry Use the graph below to answer the following questions.

13. Find the radius of each circle.
14. Are any circles congruent? How do you know?

15. Find all the common tangents for $\odot B$ and $\odot C$.
16. $\odot C$ and $\odot E$ are externally tangent. What is CE?
17. Find the equation of $\overline{CE}$.



Determine whether the given segment is tangent to $\odot K$.

18.


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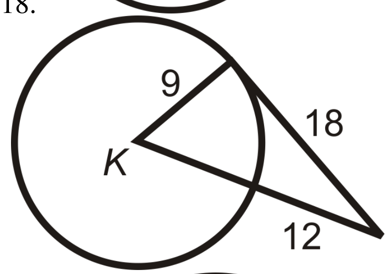

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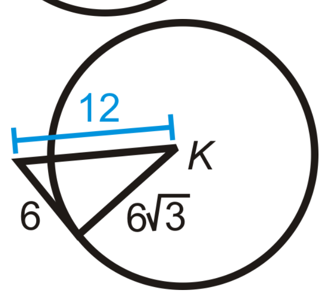
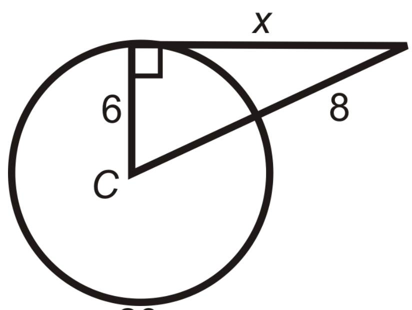


Algebra Connection Find the value of the indicated length(s) in $\odot C$. A and B are points of tangency. Simplify all radicals.



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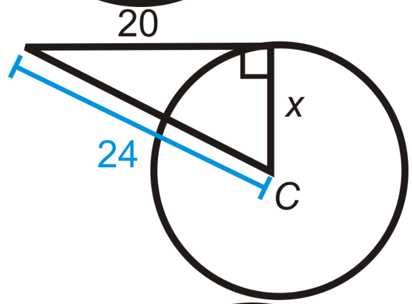

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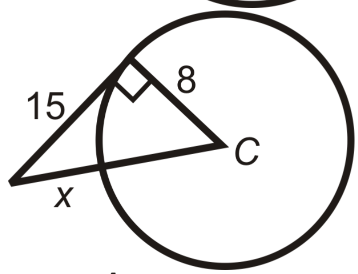

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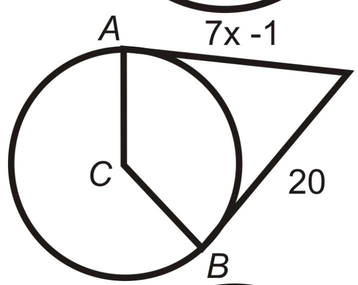

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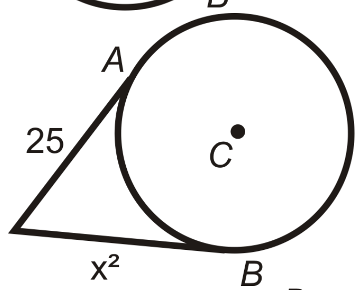

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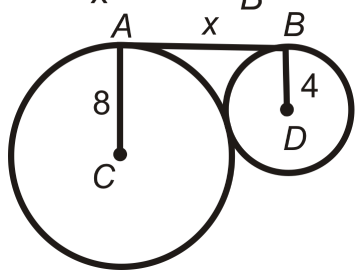
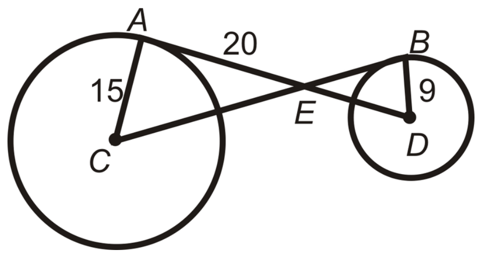


26.

27. A and B are points of tangency for $\odot C$ and $\odot D$, respectively.



a. Is $\triangle AEC \sim \triangle BED$? Why?
b. Find BC.
c. Find AD.
d. Using the trigonometric ratios, find $m\angle C$. Round to the nearest tenth of a degree.

28. Fill in the blanks in the proof of Theorem 10-2. Given: $\overline{AB}$ and $\overline{CB}$ with points of tangency at A and C. $\overline{AD}$ and $\overline{DC}$ are radii. Prove: $\overline{AB} \cong \overline{CB}$

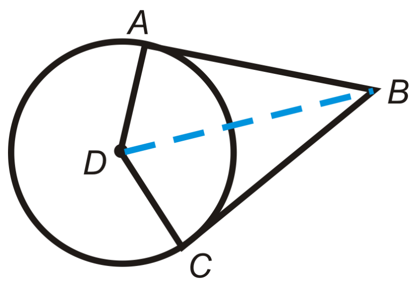


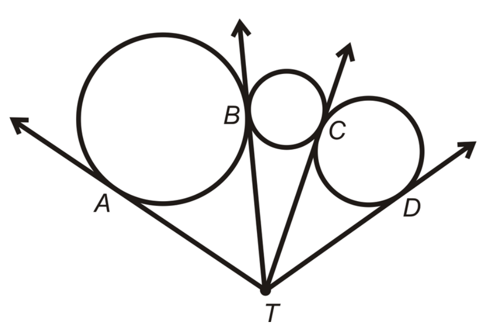
TABLE 12.1:

Statement	***Reason***
1.	
2. $\overline{AD} \cong \overline{DC}$	
3. $\overline{DA} \perp \overline{AB}$ and $\overline{DC} \perp \overline{CB}$	
4.	Definition of perpendicular lines
5.	Connecting two existing points
6. $\triangle ADB$ and $\triangle DCB$ are right triangles	
7. $\overline{DB} \cong \overline{DB}$	
8. $\triangle ABD \cong \triangle CBD$	
9. $\overline{AB} \cong \overline{CB}$	

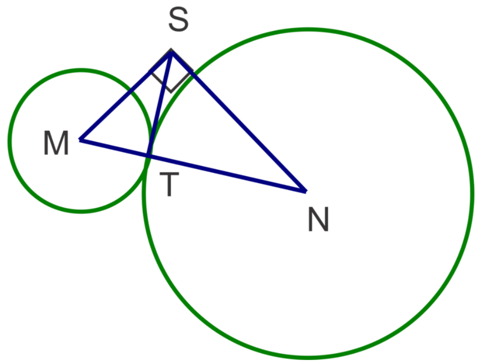
29. From the above proof, we can also conclude (fill in the blanks):

a. $ABCD$ is a _____________ (type of quadrilateral).
b. The line that connects the ___________ and the external point B _________ $\angle ADC$ and $\angle ABC$.

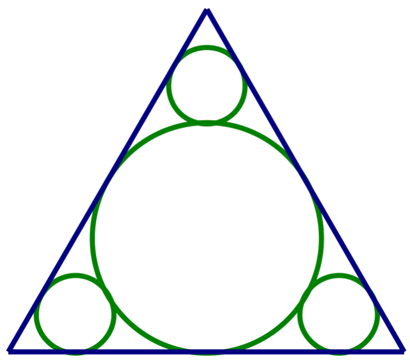
30. Points A, B, C, and D are all points of tangency for the three tangent circles. *Explain* why $\overline{AT} \cong \overline{BT} \cong \overline{CT} \cong \overline{DT}$.



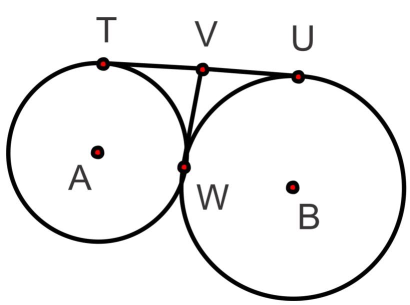
31. Circles tangent at T are centered at M and N. $\overline{ST}$ is tangent to both circles at T. Find the radius of the smaller circle if $\overline{SN} \perp \overline{SM}$, $SM = 22, TN = 25$ and $m\angle SNT = 40^\circ$.



32. Four circles are arranged inside an equilateral triangle as shown. If the triangle has sides equal to 16 cm, what is the radius of the bigger circle? What are the radii of the smaller circles?



33. Circles centered at A and B are tangent at W. Explain why A, B and W are collinear. $\overline{TU}$ is a common external tangent to the two circles. $\overline{VW}$ is tangent to both circles. Justify the fact that $\overline{TV} \cong \overline{VU} \cong \overline{VW}$.



Review Queue Answers

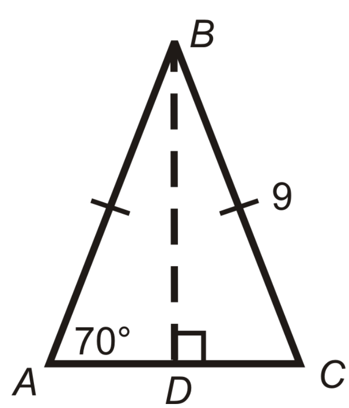
a. $y = -2x + 3$
b. $y = \frac{1}{3}x$
c. $y = -3x - 13$

12.2 Properties of Arcs

Learning Objectives

- Define and measure central angles in circles.
- Define minor arcs and major arcs.

Review Queue



1. What kind of triangle is $\triangle ABC$?
2. How does $\overline{BD}$ relate to $\triangle ABC$?
3. Find $m\angle ABC$ and $m\angle ABD$.

Round to the nearest tenth.

4. Find AD.
5. Find AC.

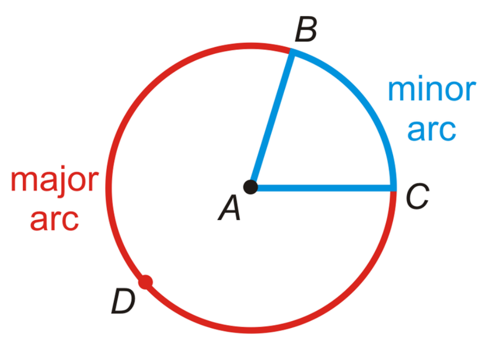
Know What? The Ferris wheel to the right has equally spaced seats, such that the central angle is 20°. How many seats are there? Why do you think it is important to have equally spaced seats on a Ferris wheel?

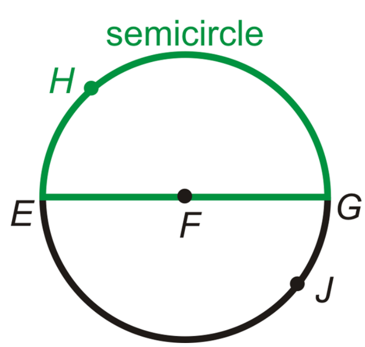
If the radius of this Ferris wheel is 25 ft., how far apart are two adjacent seats? Round your answer to the nearest tenth. *The shortest distance between two points is a straight line.*

Central Angles & Arcs

Central Angle: The angle formed by two radii of the circle with its vertex at the center of the circle.

In the picture to the right, the central angle would be $\angle BAC$. Every central angle divides a circle into two ***arcs.*** In this case the arcs are $\widehat{BC}$ and $\widehat{BDC}$. Notice the $\frown$ above the letters. To label an arc, always use this curve above the letters. Do not confuse $\overline{BC}$ and $\widehat{BC}$.





Arc: A section of the circle.

If D was not on the circle, we would not be able to tell the difference between $\widehat{BC}$ and $\widehat{BDC}$. There are 360° in a circle, where a semicircle is half of a circle, or 180°. $m\angle EFG = 180^\circ$, because it is a straight angle, so $m\widehat{EHG} = 180^\circ$ and $m\widehat{EJG} = 180^\circ$.

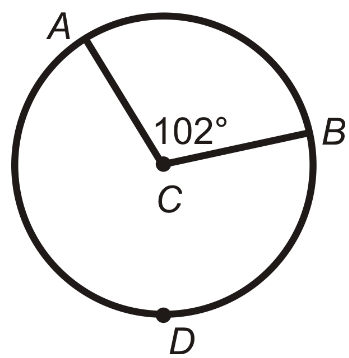
Semicircle: An arc that measures 180°.

Minor Arc: An arc that is less than 180°.

Major Arc: An arc that is greater than 180°. ***Always*** use 3 letters to label a major arc.

An arc can be measured in degrees or in a linear measure (cm, ft, etc.). In this chapter we will use degree measure. ***The measure of the minor arc is the same as the measure of the central angle*** that corresponds to it. The measure of the major arc equals to 360° minus the measure of the minor arc. In order to prevent confusion, major arcs are always named with three letters; the letters that denote the endpoints of the arc and any other point on the major arc. When referring to the measure of an arc, always place an "*m*" in from of the label.

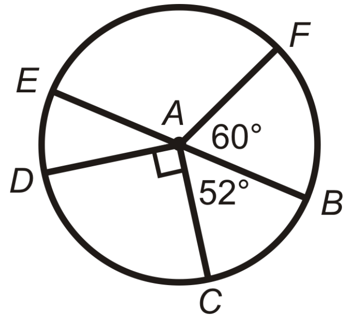
Example 1: Find $m\widehat{AB}$ and $m\widehat{ADB}$ in $\odot C$.



Solution: $m\widehat{AB}$ is the same as $m\angle ACB$. So, $m\widehat{AB} = 102^\circ$. The measure of $m\widehat{ADB}$, which is the major arc, is equal to 360° minus the minor arc.

$m\widehat{ADB} = 360^\circ - m\widehat{AB} = 360^\circ - 102^\circ = 258^\circ$

Example 2: Find the measures of the arcs in $\odot A$. $\overline{EB}$ is a diameter.

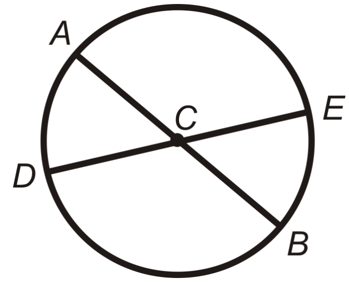


Solution: Because $\overline{EB}$ is a diameter, $m\angle EAB = 180^\circ$. Each arc is the same as its corresponding central angle.

$$m\widehat{BF} = m\angle FAB = 60^\circ$$
$$m\widehat{EF} = m\angle EAF = 120^\circ \quad \rightarrow m\angle EAB - m\angle FAB$$
$$m\widehat{ED} = m\angle EAD = 38^\circ \quad \rightarrow m\angle EAB - m\angle BAC - m\angle CAD$$
$$m\widehat{DC} = m\angle DAC = 90^\circ$$
$$m\widehat{BC} = m\angle BAC = 52^\circ$$

Congruent Arcs: Two arcs are congruent if their central angles are congruent.

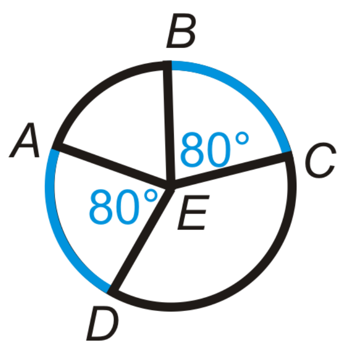
Example 3: List all the congruent arcs in $\odot C$ below. $\overline{AB}$ and $\overline{DE}$ are diameters.



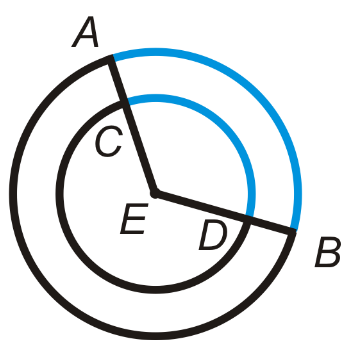
Solution: From the picture, we see that $\angle ACD$ and $\angle ECB$ are vertical angles. $\angle DCB$ and $\angle ACE$ are also vertical angles. Because all vertical angles are equal and these four angles are all central angles, we know that $\widehat{AD} \cong \widehat{EB}$ and $\widehat{AE} \cong \widehat{DB}$.

Example 4: Are the blue arcs congruent? Explain why or why not.

a)



b)



Solution: In part a, $\widehat{AD} \cong \widehat{BC}$ because they have the same central angle measure. In part b, the two arcs do have the same measure, but are not congruent because the circles are not congruent.

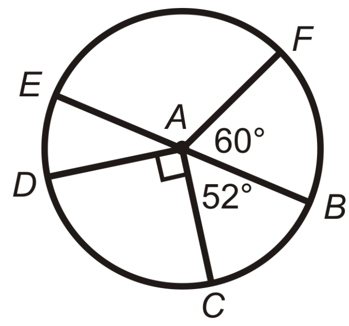
Arc Addition Postulate

Just like the Angle Addition Postulate and the Segment Addition Postulate, there is an Arc Addition Postulate. It is very similar.

Arc Addition Postulate: The measure of the arc formed by two adjacent arcs is the sum of the measures of the two arcs.

Using the picture from Example 3, we would say $m\widehat{AE} + m\widehat{EB} = m\widehat{AEB}$.

Example 5: Reusing the figure from Example 2, find the measure of the following arcs in $\odot A$. $\overline{EB}$ is a diameter.



a) $m\widehat{FED}$

b) $m\widehat{CDF}$

c) $m\widehat{BD}$

d) $m\widehat{DFC}$

Solution: Use the Arc Addition Postulate.

a) $m\widehat{FED} = m\widehat{FE} + m\widehat{ED} = 120^\circ + 38^\circ = 158^\circ$

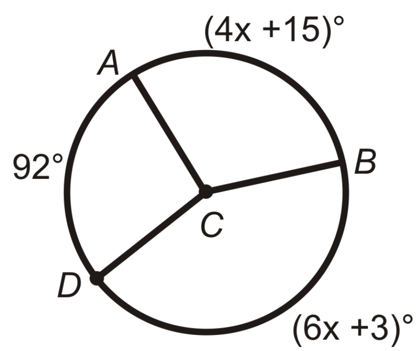
We could have labeled $\widehat{FED}$ as $\widehat{FD}$ because it is less than 180°.

b) $m\widehat{CDF} = m\widehat{CD} + m\widehat{DE} + m\widehat{EF} = 90^\circ + 38^\circ + 120^\circ = 248^\circ$

c) $m\widehat{BD} = m\widehat{BC} + m\widehat{CD} = 52^\circ + 90^\circ = 142^\circ$

d) $m\widehat{DFC} = 38^\circ + 120^\circ + 60^\circ + 52^\circ = 270^\circ$ or $m\widehat{DFC} = 360^\circ - m\widehat{CD} = 360^\circ - 90^\circ = 270^\circ$

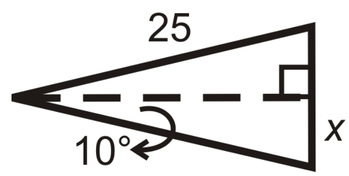
Example 6: ***Algebra Connection*** Find the value of x for $\odot C$ below.



Solution: There are 360° in a circle. Let's set up an equation.

$$\begin{aligned} m\widehat{AB} + m\widehat{AD} + m\widehat{DB} &= 360^\circ \\ (4x+15)^\circ + 92^\circ + (6x+3)^\circ &= 360^\circ \\ 10x + 110^\circ &= 360^\circ \\ 10x &= 250^\circ \\ x &= 25^\circ \end{aligned}$$

Know What? Revisited Because the seats are 20° apart, there will be $\frac{360^\circ}{20^\circ} = 18$ seats. It is important to have the seats evenly spaced for balance. To determine how far apart the adjacent seats are, use the triangle to the right. We will need to use sine to find x and then multiply it by 2.

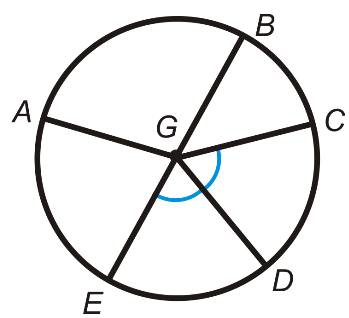


$$\sin 10^\circ = \frac{x}{25}$$
$$x = 25 \sin 10^\circ = 4.3\ ft.$$

The total distance apart is 8.6 feet.

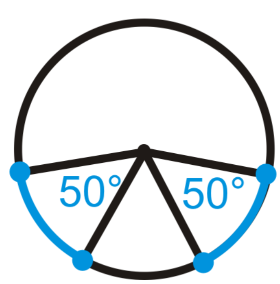
Review Questions

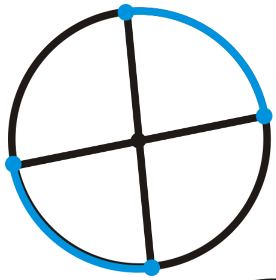
Determine if the arcs below are a minor arc, major arc, or semicircle of $\odot G$. $\overline{EB}$ is a diameter.



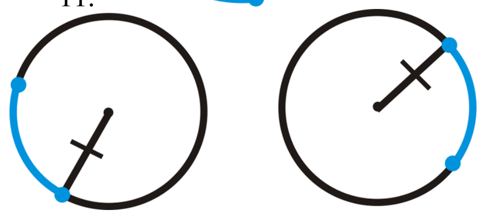
1. $\widehat{AB}$
2. $\widehat{ABD}$
3. $\widehat{BCE}$
4. $\widehat{CAE}$
5. $\widehat{ABC}$
6. $\widehat{EAB}$
7. Are there any congruent arcs? If so, list them.
8. If $m\widehat{BC} = 48^\circ$, find $m\widehat{CD}$.
9. Using #8, find $m\widehat{CAE}$.

Determine if the blue arcs are congruent. If so, state why.

10. 

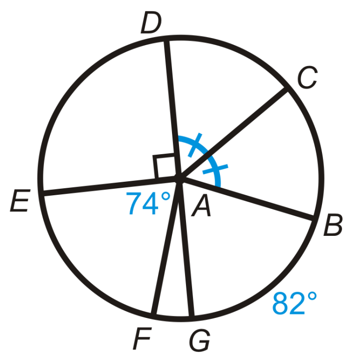


11.



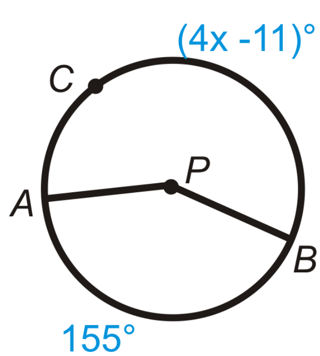
12.

Find the measure of the indicated arcs or central angles in $\odot A$. $\overline{DG}$ is a diameter.

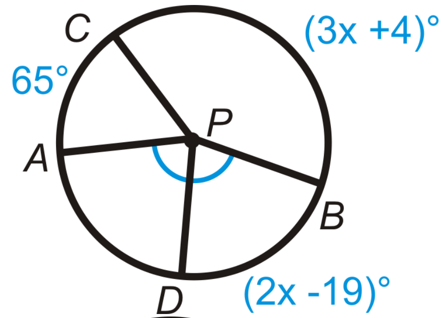


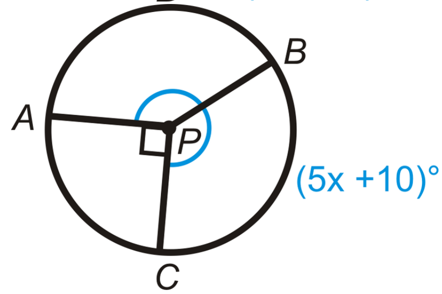
13. $\widehat{DE}$
14. $\widehat{DC}$
15. $\angle GAB$
16. $\widehat{FG}$
17. $\widehat{EDB}$
18. $\angle EAB$
19. $\widehat{DCF}$
20. $\widehat{DBE}$

Algebra Connection Find the measure of x in $\odot P$.

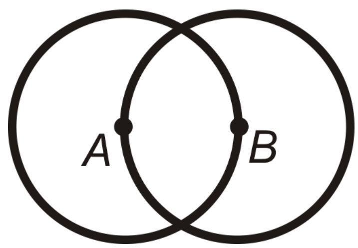


21.

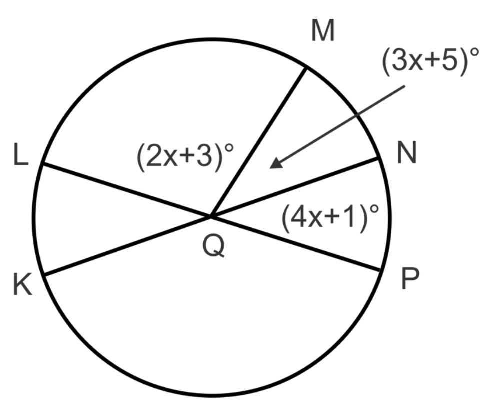
22. 

23. 

24. What can you conclude about $\odot A$ and $\odot B$?

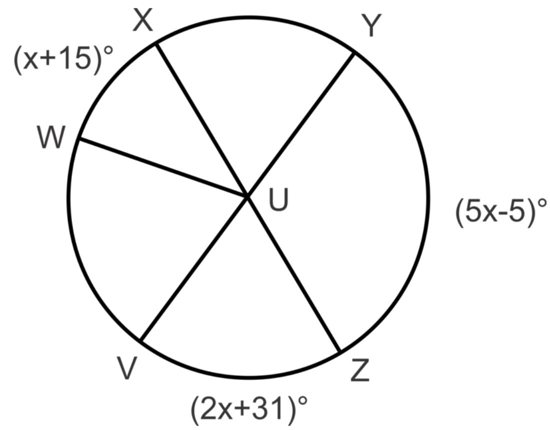


Use the diagram below to find the measures of the indicated arcs in problems 25-30.



25. $m\widehat{MN}$
26. $m\widehat{LK}$
27. $m\widehat{MP}$
28. $m\widehat{MK}$
29. $m\widehat{NPL}$
30. $m\widehat{LKM}$

Use the diagram below to find the measures indicated in problems 31-36.



31. $m\angle VUZ$
32. $m\angle YUZ$
33. $m\angle WUV$
34. $m\angle XUV$
35. $m\widehat{YWZ}$
36. $m\widehat{WYZ}$

Review Queue Answers

a. isosceles
b. $\overline{BD}$ is the angle bisector of $\angle ABC$ and the perpendicular bisector of $\overline{AC}$.
c. $m\angle ABC = 40^\circ, m\angle ABD = 25^\circ$
d. $\cos 70^\circ = \frac{AD}{9} \rightarrow AD = 9 \cdot \cos 70^\circ = 3.1$
e. $AC = 2 \cdot AD = 2 \cdot 3.1 = 6.2$

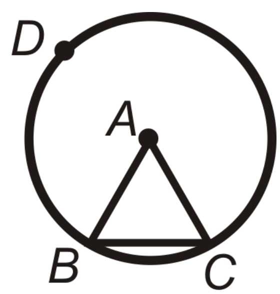
12.3 Properties of Chords

Learning Objectives

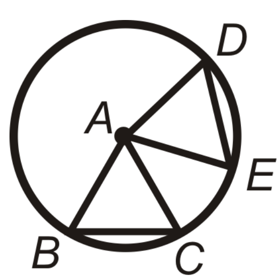
- Find the lengths of chords in a circle.
- Discover properties of chords and arcs.

Review Queue

a. Draw a chord in a circle.
b. Draw a diameter in the circle from #1. Is a diameter a chord?
c. $\triangle ABC$ is an equilateral triangle in $\odot A$. Find $m\widehat{BC}$ and $m\widehat{BDC}$.



d. $\triangle ABC$ and $\triangle ADE$ are equilateral triangles in $\odot A$. List a pair of congruent arcs and chords.



Know What? To the right is the Gran Teatro Falla, in Cadiz, Andalucía, Spain. This theater was built in 1905 and hosts several plays and concerts. It is an excellent example of circles in architecture. Notice the five windows, $A-E$. $\odot A \cong \odot E$ and $\odot B \cong \odot C \cong \odot D$. Each window is topped with a 240° arc. The gold chord in each circle connects the rectangular portion of the window to the circle. Which chords are congruent? How do you know?

Recall from the first section, that a chord is a line segment whose endpoints are on a circle. A diameter is the longest chord in a circle. There are several theorems that explore the properties of chords.

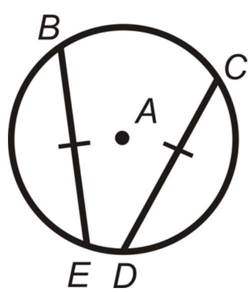
Congruent Chords & Congruent Arcs

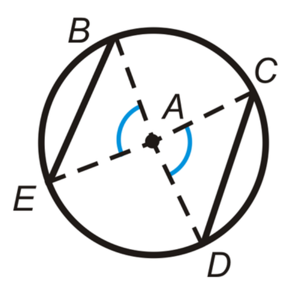
From #4 in the Review Queue above, we noticed that $\overline{BC} \cong \overline{DE}$ and $\widehat{BC} \cong \widehat{DE}$. This leads to our first theorem.

Theorem 10-3: In the same circle or congruent circles, minor arcs are congruent if and only if their corresponding chords are congruent.

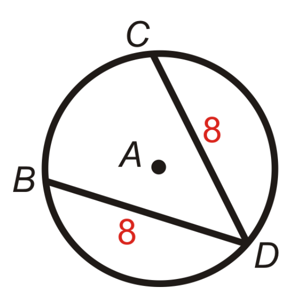
Notice the "if and only if" in the middle of the theorem. This means that Theorem 10-3 is a biconditional statement. Taking this theorem one step further, any time two central angles are congruent, the chords and arcs from the endpoints of the sides of the central angles are also congruent.

In both of these pictures, $\overline{BE} \cong \overline{CD}$ and $\widehat{BE} \cong \widehat{CD}$. In the second picture, we have $\triangle BAE \cong \triangle CAD$ because the central angles are congruent and $\overline{BA} \cong \overline{AC} \cong \overline{AD} \cong \overline{AE}$ because they are all radii (SAS). By CPCTC, $\overline{BE} \cong \overline{CD}$.





Example 1: Use $\odot A$ to answer the following.



a) If $m\widehat{BD} = 125^\circ$, find $m\widehat{CD}$.

b) If $m\widehat{BC} = 80^\circ$, find $m\widehat{CD}$.

Solution:

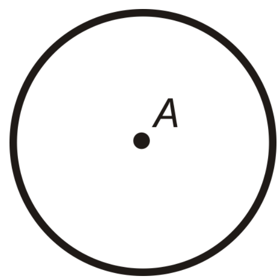
a) From the picture, we know $BD = CD$. Because the chords are equal, the arcs are too. $m\widehat{CD} = 125^\circ$.

b) To find $m\widehat{CD}$, subtract 80° from 360° and divide by 2. $m\widehat{CD} = \frac{360^\circ - 80^\circ}{2} = \frac{280^\circ}{2} = 140^\circ$

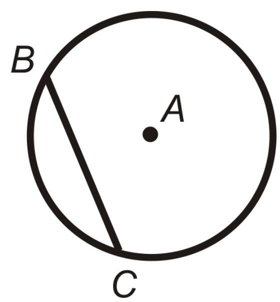
Investigation 9-2: Perpendicular Bisector of a Chord

Tools Needed: paper, pencil, compass, ruler

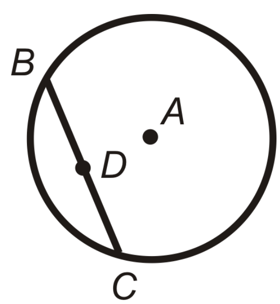
a. Draw a circle. Label the center A.



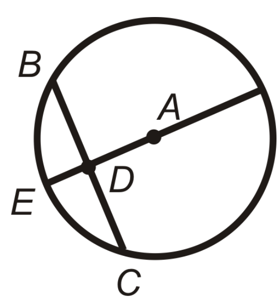
b. Draw a chord in $\odot A$. Label it $\overline{BC}$.



c. Find the midpoint of $\overline{BC}$ by using a ruler. Label it D.



d. Connect A and D to form a diameter. How does $\overline{AD}$ relate to the chord, $\overline{BC}$?

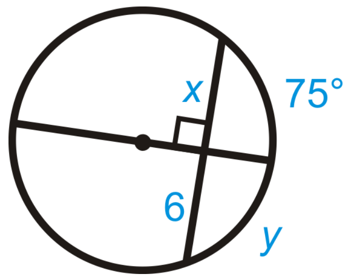


Theorem 10-4: The perpendicular bisector of a chord is also a diameter.

In the picture to the left, $\overline{AD} \perp \overline{BC}$ and $\overline{BD} \cong \overline{DC}$. From this theorem, we also notice that $\overline{AD}$ also bisects the corresponding arc at E, so $\widehat{BE} \cong \widehat{EC}$.

Theorem 10-5: If a diameter is perpendicular to a chord, then the diameter bisects the chord and its corresponding arc.

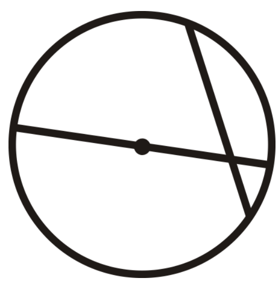
Example 2: Find the value of x and y.



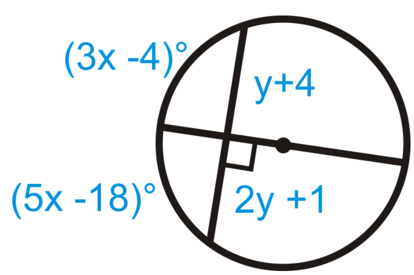
Solution: The diameter here is also perpendicular to the chord. From Theorem 10-5, $x = 6$ and $y = 75^\circ$.

Example 3: Is the converse of Theorem 10-4 true?

Solution: The converse of Theorem 10-4 would be: A diameter is also the perpendicular bisector of a chord. This is not a true statement, see the counterexample to the right.



Example 4: ***Algebra Connection*** Find the value of x and y.



Solution: Because the diameter is perpendicular to the chord, it also bisects the chord and the arc. Set up an equation for x and y.

$$\begin{aligned}(3x-4)^\circ &= (5x-18)^\circ \\ 14^\circ &= 2x \\ 7^\circ &= x\end{aligned}$$

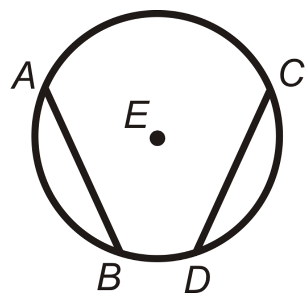
$$\begin{aligned}y+4 &= 2y+1 \\ 3 &= y\end{aligned}$$

Equidistant Congruent Chords

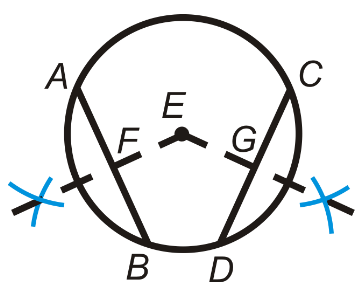
Investigation 9-3: Properties of Congruent Chords

Tools Needed: pencil, paper, compass, ruler

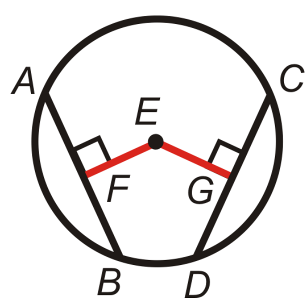
a. Draw a circle with a radius of 2 inches and two chords that are both 3 inches. Label as in the picture to the right. *This diagram is drawn to scale.*



b. From the center, draw the perpendicular segment to $\overline{AB}$ and $\overline{CD}$. You can either use your ruler, a protractor or Investigation 3-2 (Constructing a Perpendicular Line through a Point not on the line. We will show arc marks for Investigation 3-2.



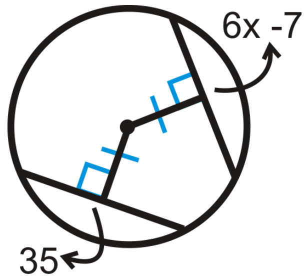
c. Erase the arc marks and lines beyond the points of intersection, leaving $\overline{FE}$ and $\overline{EG}$. Find the measure of these segments. What do you notice?



Theorem 10-6: In the same circle or congruent circles, two chords are congruent if and only if they are equidistant from the center.

Recall that two lines are equidistant from the same point if and only if the shortest distance from the point to the line is congruent. The shortest distance from any point to a line is the perpendicular line between them. In this theorem, the fact that $FE = EG$ means that $\overline{AB}$ and $\overline{CD}$ are equidistant to the center and $\overline{AB} \cong \overline{CD}$.

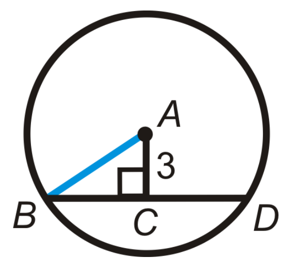
Example 5: ***Algebra Connection*** Find the value of x.



Solution: Because the distance from the center to the chords is congruent and perpendicular to the chords, then the chords are equal.

$$\begin{aligned} 6x-7&=35 \\ 6x&=42 \\ x&=7 \end{aligned}$$

Example 6: $BD = 12$ and $AC = 3$ in $\odot A$. Find the radius and $m\widehat{BD}$.



Solution: First find the radius. In the picture, $\overline{AB}$ is a radius, so we can use the right triangle $\triangle ABC$, such that $\overline{AB}$ is the hypotenuse. From 10-5, $BC = 6$.

$$\begin{aligned} 3^2+6^2&=AB^2 \\ 9+36&=AB^2 \\ AB&=\sqrt{45}=3\sqrt{5} \end{aligned}$$

In order to find $m\widehat{BD}$, we need the corresponding central angle, $\angle BAD$. We can find half of $\angle BAD$ because it is an acute angle in $\triangle ABC$. Then, multiply the measure by 2 for $m\widehat{BD}$.

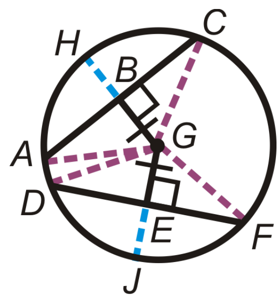
$$\begin{aligned} \tan^{-1}\left(\frac{6}{3}\right)&=m\angle BAC \\ m\angle BAC&\approx 63.43^\circ \end{aligned}$$

This means that $m\angle BAD \approx 126.9^\circ$ and $m\widehat{BD} \approx 126.9^\circ$ as well.

Know What? Revisited In the picture, the chords from $\odot A$ and $\odot E$ are congruent and the chords from $\odot B, \odot C$, and $\odot D$ are also congruent. We know this from Theorem 10-3. All five chords are not congruent because all five circles are not congruent, even though the central angle for the circles is the same.

Review Questions

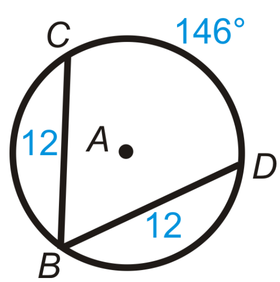
1. Two chords in a circle are perpendicular and congruent. Does one of them have to be a diameter? Why or why not? Fill in the blanks.



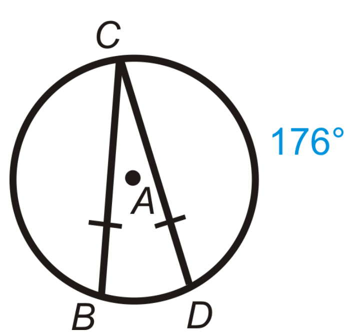
2. ___ $\cong \overline{DF}$
3. $\widehat{AC} \cong$ ___
4. $\widehat{DJ} \cong$ ___
5. ___ $\cong \overline{EJ}$
6. $\angle AGH \cong$ ___
7. $\angle DGF \cong$ ___
8. List all the congruent radii in $\odot G$.

Find the value of the indicated arc in $\odot A$.

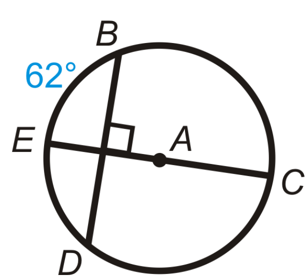
9. $m\widehat{BC}$



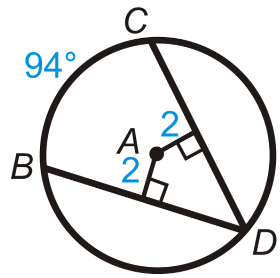
10. $m\widehat{BD}$



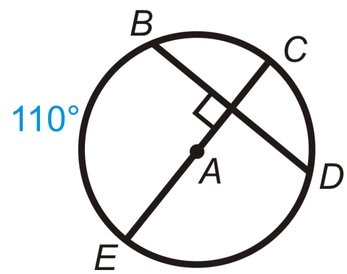
11. $m\widehat{BC}$



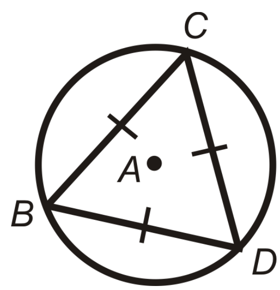
12. $m\widehat{BD}$



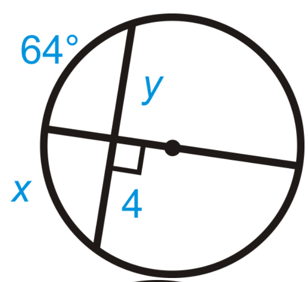
13. $m\widehat{BD}$



14. $m\widehat{BD}$



Algebra Connection Find the value of x and/or y.



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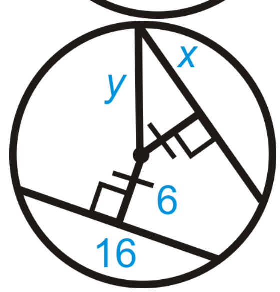

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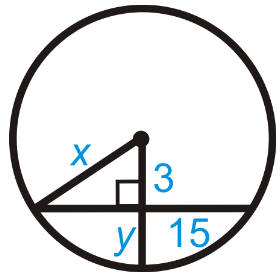
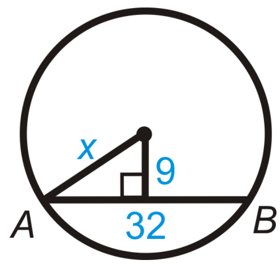
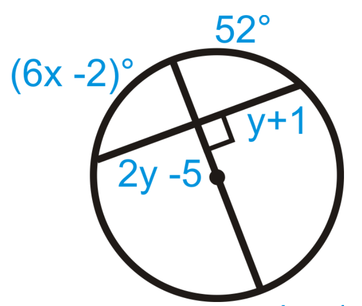
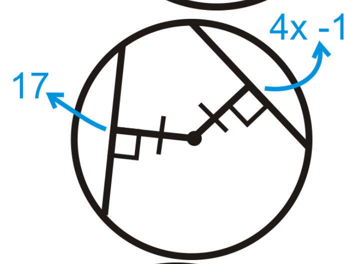
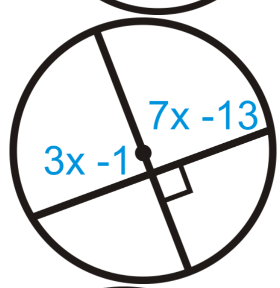
17. 

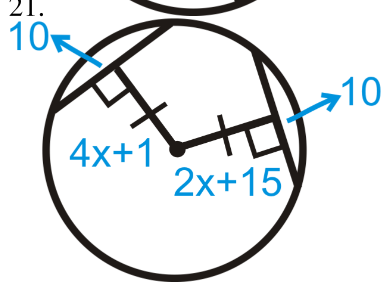
18. $AB = 32$

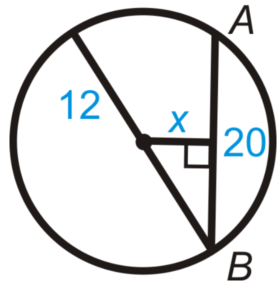


19. 

20. 

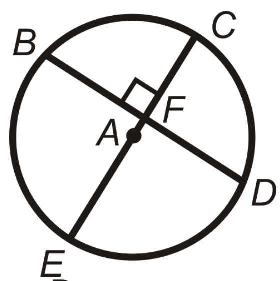
21. 

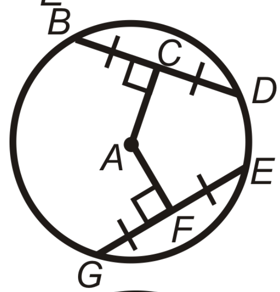
22. 

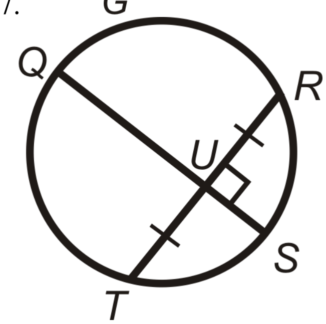
23. 

24. Find $m\widehat{AB}$ in Question 18. Round your answer to the nearest tenth of a degree.
25. Find $m\widehat{AB}$ in Question 23. Round your answer to the nearest tenth of a degree.

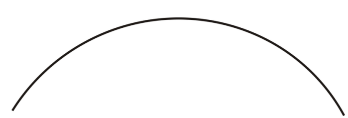
In problems 26-28, what can you conclude about the picture? State a theorem that justifies your answer. You may assume that A is the center of the circle.

26. 

27. 

28. 

29. Trace the arc below onto your paper then follow the steps to locate the center using a compass and straightedge.



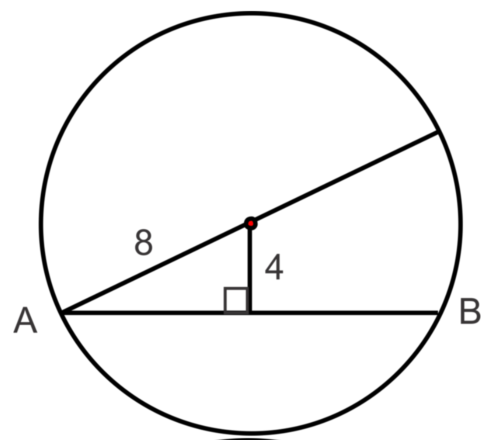
   a. Use your straightedge to make a chord in the arc.
   b. Use your compass and straightedge to construct the perpendicular bisector of this chord.
   c. Repeat steps a and b so that you have two chords and their perpendicular bisectors.
   d. What is the significance of the point where the perpendicular bisectors intersect?
   e. Verify your answer to part d by using the point and your compass to draw the rest of the circle.

30. ***Algebra Connection*** Let's repeat what we did in problem 29 using coordinate geometry skills. Given the points $A(-3,5), B(5,5)$ and $C(4,-2)$ on the circle (an arc could be drawn through these points from A to C). The following steps will walk you through the process to find the equation of the perpendicular bisector of

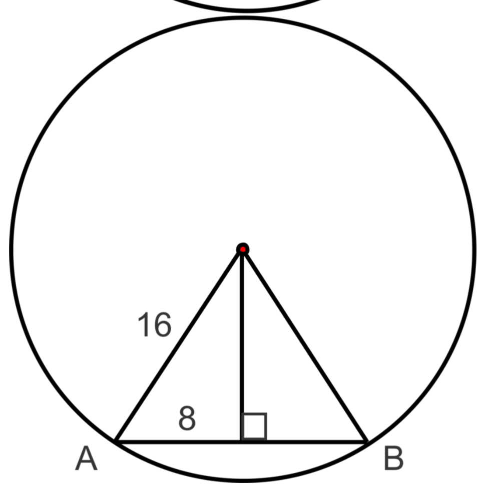
a chord, and use two of these perpendicular bisectors to locate the center of the circle. Let's first find the perpendicular bisector of chord $\overline{AB}$.

a. Since the perpendicular bisector passes through the midpoint of a segment we must first find the midpoint between A and B.
b. Now the perpendicular line must have a slope that is the opposite reciprocal of the slope of $\overleftrightarrow{AB}$. Find the slope of $\overleftrightarrow{AB}$ and then its opposite reciprocal.
c. Finally, you can write the equation of the perpendicular bisector of $\overline{AB}$ using the point you found in part a and the slope you found in part b.
d. Repeat steps a-c for chord $\overline{BC}$.
e. Now that we have the two perpendicular bisectors of the chord we can use algebra to find their intersection. Solve the system of linear equations to find the center of the circle.
f. Find the radius of the circle by finding the distance from the center (point found in part e) to any of the three given points on the circle.

31. Find the measure of $\widehat{AB}$ in each diagram below.



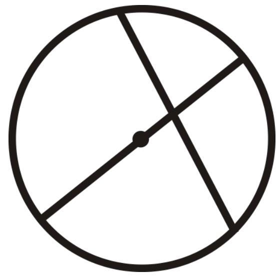
a.



b.

Review Queue Answers

1 & 2. Answers will vary



3. $m\widehat{BC} = 60^\circ, m\widehat{BDC} = 300^\circ$
4. $\overline{BC} \cong \overline{DE}$ and $\widehat{BC} \cong \widehat{DE}$

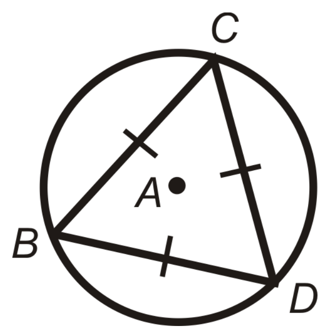
12.4 Inscribed Angles

Learning Objectives

- Find the measure of inscribed angles and the arcs they intercept.

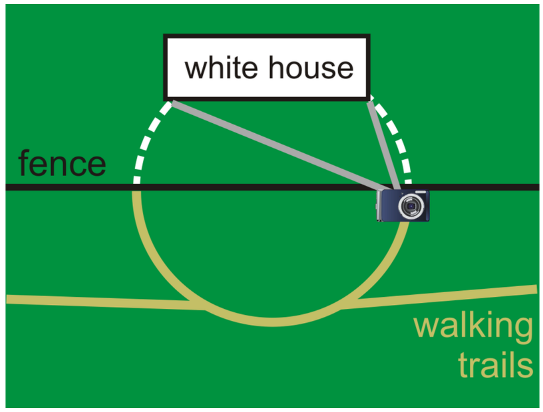
Review Queue

We are going to use #14 from the homework in the previous section.



a. What is the measure of each angle in the triangle? How do you know?
b. What do you know about the three arcs?
c. What is the measure of each arc?
d. What is the relationship between the angles in the triangles and the measure of each arc?

Know What? Your family went to Washington DC over the summer and saw the White House. The closest you can get to the White House are the walking trails on the far right. You got as close as you could (on the trail) to the fence to take a picture (you were not allowed to walk on the grass). Where else could you have taken your picture from to get the same frame of the White House? Where do you think the best place to stand would be? *Your line of sight in the camera is marked in the picture as the grey lines. The white dotted arcs do not actually exist, but were added to help with this problem.*



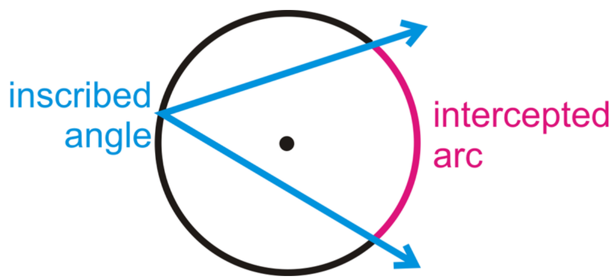
Inscribed Angles

We have discussed central angles so far in this chapter. We will now introduce another type of angle, the inscribed angle.

Inscribed Angle: An angle with its vertex is the circle and its sides contain chords.

Intercepted Arc: The arc that is on the interior of the inscribed angle and whose endpoints are on the angle.

The vertex of an inscribed angle can be anywhere on the circle as long as its sides intersect the circle to form an intercepted arc.

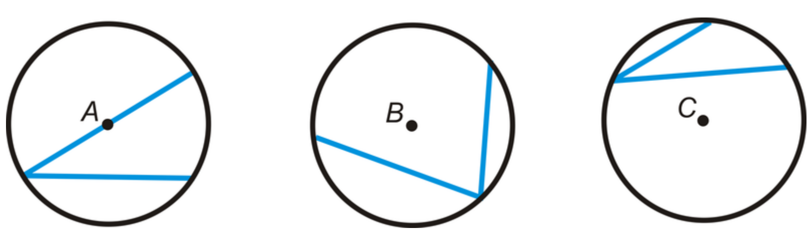


Now, we will investigation the relationship between the inscribed angle, the central angle and the arc they intercept.

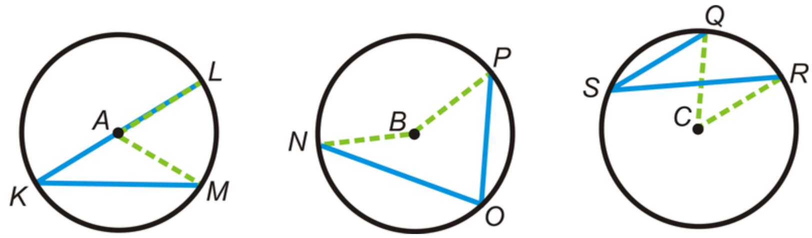
Investigation 9-4: Measuring an Inscribed Angle

Tools Needed: pencil, paper, compass, ruler, protractor

1. Draw three circles with three different inscribed angles. For $\odot A$, make one side of the inscribed angle a diameter, for $\odot B$, make B inside the angle and for $\odot C$ make C outside the angle. Try to make all the angles different sizes.



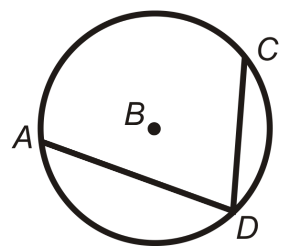
2. Using your ruler, draw in the corresponding central angle for each angle and label each set of endpoints.



3. Using your protractor measure the six angles and determine if there is a relationship between the central angle, the inscribed angle, and the intercepted arc.

$m\angle LAM =$ ____	$m\angle NBP =$ ____	$m\angle QCR =$ ____
$m\widehat{LM} =$ ____	$m\widehat{NP} =$ ____	$m\widehat{QR} =$ ____
$m\angle LKM =$ ____	$m\angle NOP =$ ____	$m\angle QSR =$ ____

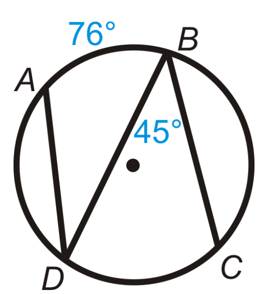
Inscribed Angle Theorem: The measure of an inscribed angle is half the measure of its intercepted arc.



In the picture, $m\angle ADC = \frac{1}{2} m\widehat{AC}$. If we had drawn in the central angle $\angle ABC$, we could also say that $m\angle ADC = \frac{1}{2} m\angle ABC$ because the measure of the central angle is equal to the measure of the intercepted arc.

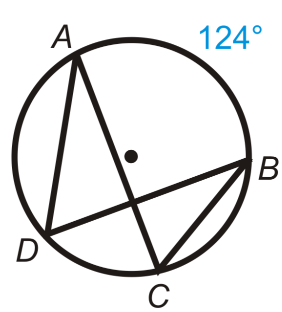
To prove the Inscribed Angle Theorem, you would need to split it up into three cases, like the three different angles drawn from Investigation 9-4. We will touch on the algebraic proofs in the review exercises.

Example 1: Find $m\widehat{DC}$ and $m\angle ADB$.



Solution: From the Inscribed Angle Theorem, $m\widehat{DC} = 2 \cdot 45^\circ = 90^\circ$. $m\angle ADB = \frac{1}{2} \cdot 76^\circ = 38^\circ$.

Example 2: Find $m\angle ADB$ and $m\angle ACB$.



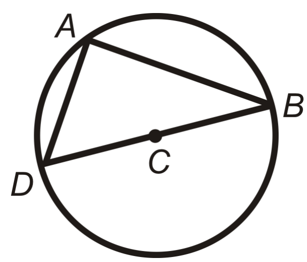
Solution: The intercepted arc for both angles is $\widehat{AB}$. Therefore, $m\angle ADB = m\angle ACB = \frac{1}{2} \cdot 124^\circ = 62^\circ$

This example leads us to our next theorem.

Theorem 9-8: Inscribed angles that intercept the same arc are congruent.

To prove Theorem 9-8, you would use the similar triangles that are formed by the chords.

Example 3: Find $m\angle DAB$ in $\odot C$.



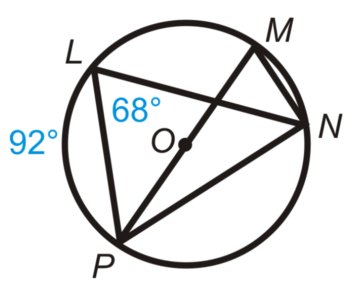
Solution: Because C is the center, $\overline{DB}$ is a diameter. Therefore, $\angle DAB$ inscribes semicircle, or 180°. $m\angle DAB = \frac{1}{2} \cdot 180^\circ = 90^\circ$.

Theorem 9-9: An angle that intercepts a semicircle is a right angle.

In Theorem 9-9 we could also say that the angle is inscribed in a semicircle. Anytime a right angle is inscribed in a circle, the endpoints of the angle are the endpoints of a diameter. Therefore, the converse of Theorem 9-9 is also true.

When the three vertices of a triangle are on the circle, like in Example 3, we say that the triangle is *inscribed* in the circle. We can also say that the circle is *circumscribed* around (or about) the triangle. Any polygon can be inscribed in a circle.

Example 4: Find $m\angle PMN, m\widehat{PN}, m\angle MNP, m\angle LNP$, and $m\widehat{LN}$.



Solution:

$m\angle PMN = m\angle PLN = 68^\circ$ by Theorem 9-8.

$m\widehat{PN} = 2 \cdot 68^\circ = 136^\circ$ from the Inscribed Angle Theorem.

$m\angle MNP = 90^\circ$ by Theorem 9-9.

$m\angle LNP = \frac{1}{2} \cdot 92^\circ = 46^\circ$ from the Inscribed Angle Theorem.

To find $m\widehat{LN}$, we need to find $m\angle LPN$. $\angle LPN$ is the third angle in $\triangle LPN$, so $68^\circ + 46^\circ + m\angle LPN = 180^\circ$. $m\angle LPN = 66^\circ$, which means that $m\widehat{LN} = 2 \cdot 66^\circ = 132^\circ$.

Inscribed Quadrilaterals

The last theorem for this section involves inscribing a quadrilateral in a circle.

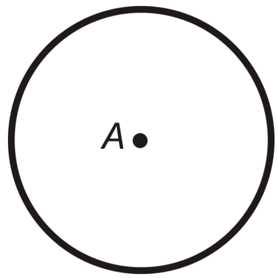
Inscribed Polygon: A polygon where every vertex is on a circle.

Note, that not every quadrilateral or polygon can be inscribed in a circle. Inscribed quadrilaterals are also called *cyclic quadrilaterals.* For these types of quadrilaterals, they must have one special property. We will investigate it here.

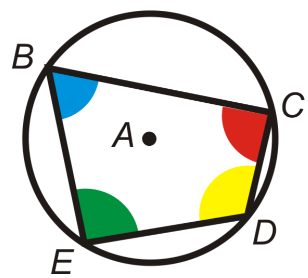
Investigation 9-5: Inscribing Quadrilaterals

Tools Needed: pencil, paper, compass, ruler, colored pencils, scissors

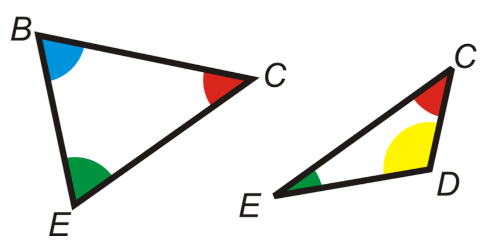
a. Draw a circle. Mark the center point A.



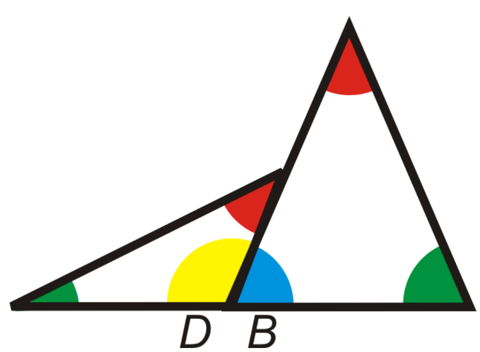
b. Place four points on the circle. Connect them to form a quadrilateral. Color the 4 angles of the quadrilateral 4 different colors.



c. Cut out the quadrilateral. Then cut the quadrilateral into two triangles, by cutting on a diagonal.



d. Line up $\angle B$ and $\angle D$ so that they are adjacent angles. What do you notice? What does this show?

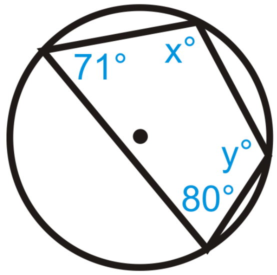


This investigation shows that the opposite angles in an inscribed quadrilateral are supplementary. By cutting the quadrilateral in half, through the diagonal, we were able to show that the other two angles (that we did not cut through) formed a linear pair when matched up.

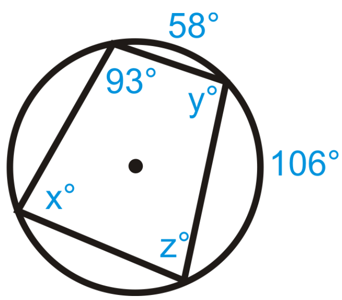
Theorem 9-10: A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary.

Example 5: Find the value of the missing variables.

a)



b)



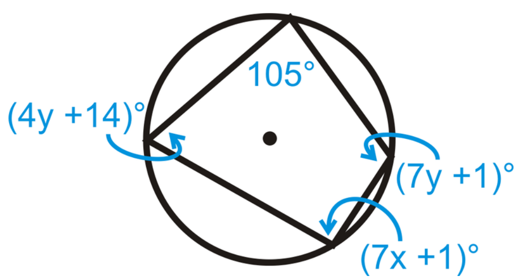
Solution:

a) $x+80^\circ = 180^\circ$ by Theorem 9-10. $x = 100^\circ$.

$y+71^\circ = 180^\circ$ by Theorem 9-10. $y = 109^\circ$.

b) It is easiest to figure out z first. It is supplementary with 93°, so $z = 87^\circ$. Second, we can find x. x is an inscribed angle that intercepts the arc $58^\circ + 106^\circ = 164^\circ$. Therefore, by the Inscribed Angle Theorem, $x = 82^\circ$. y is supplementary with x, so $y = 98^\circ$.

Example 6: ***Algebra Connection*** Find x and y in the picture below.

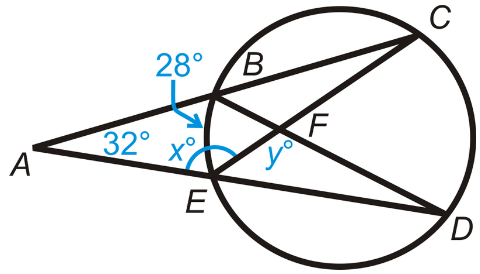


Solution: The opposite angles are supplementary. Set up an equation for x and y.

$$\begin{aligned}(7x+1)^\circ + 105^\circ &= 180^\circ \\ 7x+106^\circ &= 180^\circ \\ 7x &= 84^\circ \\ x &= 12^\circ\end{aligned}$$

$$\begin{aligned}(4y+14)^\circ + (7y+1)^\circ &= 180^\circ \\ 11y+15^\circ &= 180^\circ \\ 11y &= 165^\circ \\ y &= 15^\circ\end{aligned}$$

Example 7: Find x and y in the picture below.



Solution: To find x, use $\triangle ACE$. $m\angle ACE = 14^\circ$ because it is half of $m\widehat{BE}$ by the Inscribed Angle Theorem.

$$\begin{aligned} 32^\circ + 14^\circ + x^\circ &= 180^\circ \\ x &= 134^\circ \end{aligned}$$

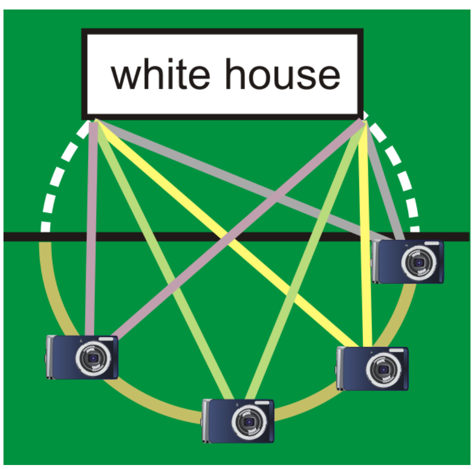
To find y, we will use $\triangle EFD$.

$$\begin{aligned} m\angle FED &= 180^\circ - x^\circ \\ m\angle FED &= 180^\circ - 134^\circ = 46^\circ \end{aligned}$$

$m\angle BDE = m\angle ACE = 14^\circ$ because they intercept the same arc, Theorem 9-8. Let's solve for y in $\triangle EFD$, using the Triangle Sum Theorem.

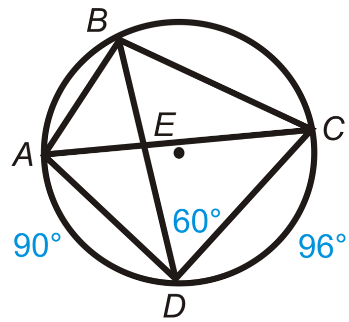
$$\begin{aligned} 46^\circ + 14^\circ + y^\circ &= 180^\circ \\ y &= 120^\circ \end{aligned}$$

Know What? Revisited You can take the picture from anywhere on the semicircular walking path. The best place to take the picture is subjective, but most would think the pale green frame, straight-on, would be the best view.



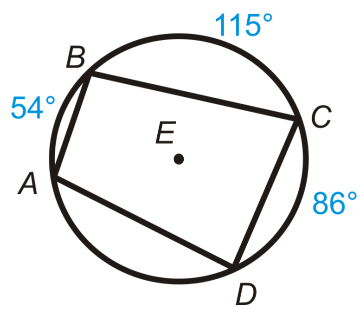
Review Questions

Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:



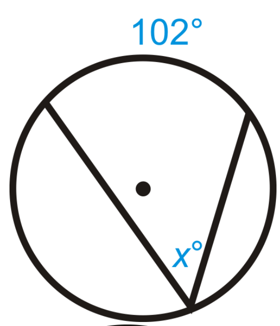
1. $m\angle DBC$
2. $m\widehat{BC}$
3. $m\widehat{AB}$
4. $m\angle ACD$
5. $m\angle ADC$
6. $m\angle ACB$

Quadrilateral $ABCD$ is inscribed in $\odot E$. Find:



7. $m\angle A$
8. $m\angle B$
9. $m\angle C$
10. $m\angle D$

Find the value of x and/or y in $\odot A$.



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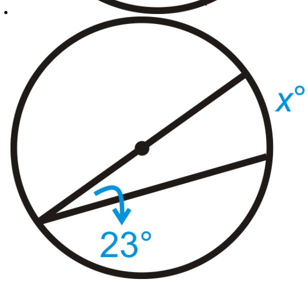

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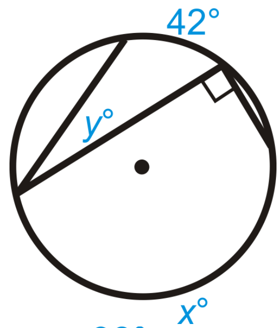

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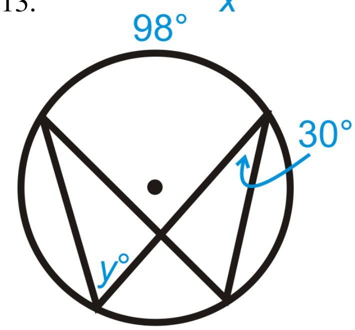

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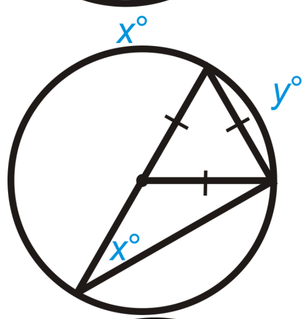
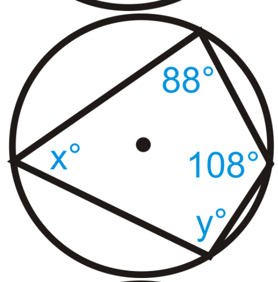


16.



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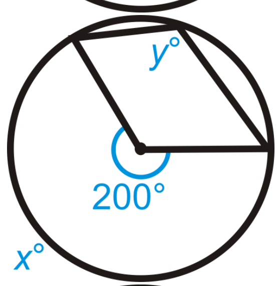

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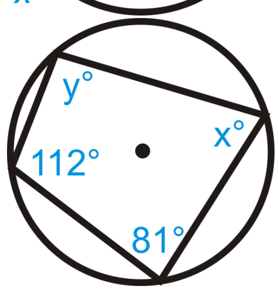
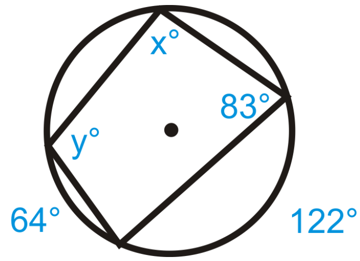
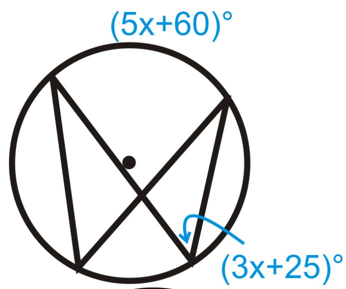
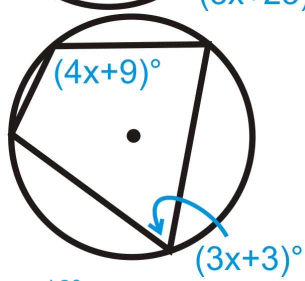
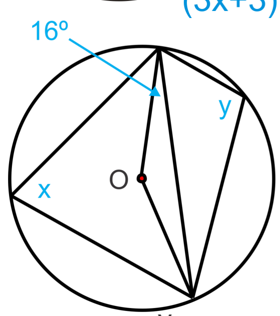


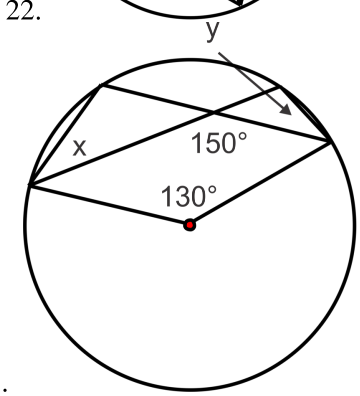
19. 

Algebra Connection Solve for the variables.

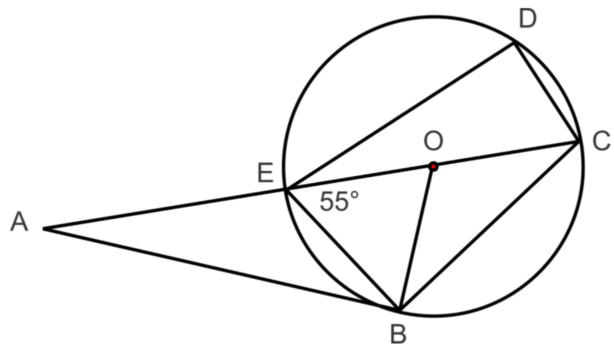
20. 

21. 

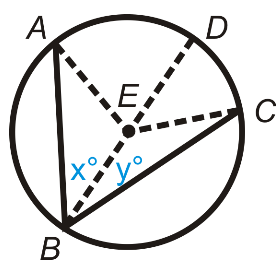
22. 

23. 

Use the diagram below to find the measures of the indicated angles and arcs in problems 24-28.



24. $m\angle EBO$
25. $m\angle EOB$
26. $m\widehat{BC}$
27. $m\angle ABO$
28. $m\angle A$
29. $m\angle EDC$
30. Fill in the blanks of one proof of the Inscribed Angle Theorem.

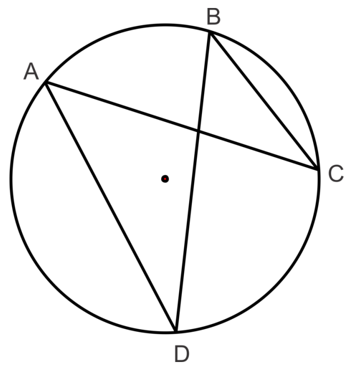


Given: Inscribed $\angle ABC$ and diameter $\overline{BD}$ Prove: $m\angle ABC = \frac{1}{2}m\widehat{AC}$

TABLE 12.2:

Statement	**Reason**
1. Inscribed $\angle ABC$ and diameter $\overline{BD}$ $m\angle ABE = x^\circ$ and $m\angle CBE = y^\circ$	
2. $x^\circ + y^\circ = m\angle ABC$	
3.	All radii are congruent
4.	Definition of an isosceles triangle
5. $m\angle EAB = x^\circ$ and $m\angle ECB = y^\circ$	
6. $m\angle AED = 2x^\circ$ and $m\angle CED = 2y^\circ$	
7. $m\widehat{AD} = 2x^\circ$ and $m\widehat{DC} = 2y^\circ$	
8.	Arc Addition Postulate
9. $m\widehat{AC} = 2x^\circ + 2y^\circ$	
10.	Distributive PoE
11. $m\widehat{AC} = 2m\angle ABC$	
12. $m\angle ABC = \frac{1}{2}m\widehat{AC}$	

31. Use the diagram below to write a proof of Theorem 9-8.



32. Suppose that $\overline{AB}$ is a diameter of a circle centered at O, and C is any other point on the circle. Draw the line through O that is parallel to $\overline{AC}$, and let D be the point where it meets $\widehat{BC}$. Prove that D is the midpoint of $\widehat{BC}$.

Review Queue Answers

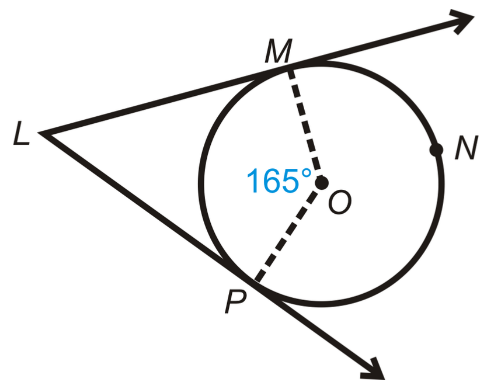
a. 60°, it is an equilateral triangle.
b. They are congruent because the chords are congruent.
c. $\frac{360^\circ}{3} = 120^\circ$
d. The arcs are double each angle.

12.5 Angles of Chords, Secants, and Tangents

Learning Objectives

- Find the measures of angles formed by chords, secants, and tangents.

Review Queue



a. What is $m\angle OML$ and $m\angle OPL$? How do you know?
b. Find $m\angle MLP$.
c. Find $m\widehat{MNP}$.
d. Find $\frac{m\widehat{MNP}-m\widehat{MP}}{2}$. What is it the same as?

Know What? The sun's rays hit the Earth such that the tangent rays determine when daytime and night time are. The time and Earth's rotation determine when certain locations have sun. If the arc that is exposed to sunlight is 178°, what is the angle at which the sun's rays hit the earth (x°)?

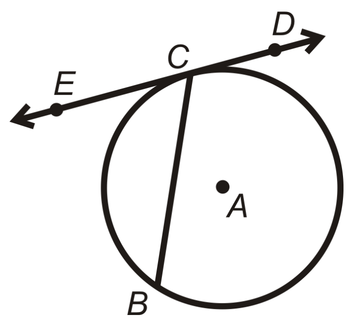
Angle

When an angle is on a circle, the vertex is on the circumference of the circle. One type of angle *on* a circle is the inscribed angle, from the previous section. Recall that ***an inscribed angle is formed by two chords and is half the measure of the intercepted arc.*** Another type of angle *on* a circle is one formed by a tangent and a chord.

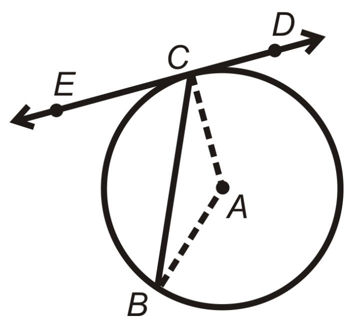
Investigation 9-6: The Measure of an Angle formed by a Tangent and a Chord

Tools Needed: pencil, paper, ruler, compass, protractor

a. Draw $\odot A$ with chord $\overline{BC}$ and tangent line $\overleftrightarrow{ED}$ with point of tangency C.



b. Draw in central angle $\angle CAB$. Then, using your protractor, find $m\angle CAB$ and $m\angle BCE$.



c. Find $m\widehat{BC}$ (the minor arc). How does the measure of this arc relate to $m\angle BCE$?

What other angle that you have learned about is this type of angle similar to?

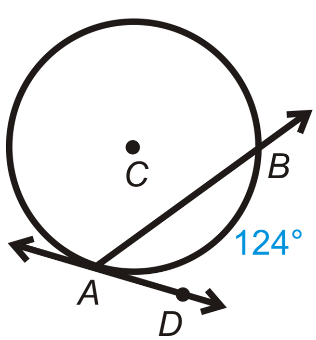
This investigation proves Theorem 9-11.

Theorem 9-11: The measure of an angle formed by a chord and a tangent that intersect on the circle is half the measure of the intercepted arc.

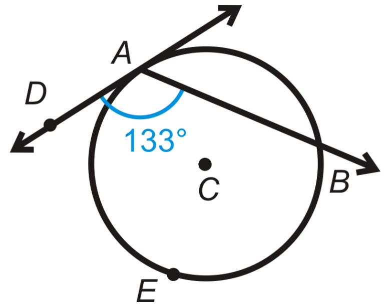
From Theorem 9-11, we now know that there are two types of angles that are half the measure of the intercepted arc; an inscribed angle and an angle formed by a chord and a tangent. Therefore, ***any angle with its vertex on a circle will be half the measure of the intercepted arc***.

Example 1: Find:

a) $m\angle BAD$



b) $m\widehat{AEB}$

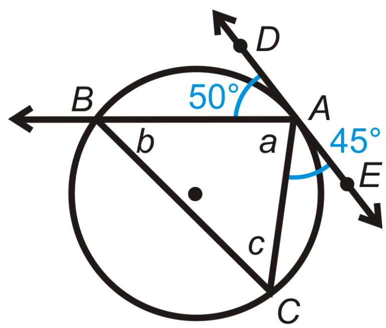


Solution: Use Theorem 9-11.

a) $m\angle BAD = \frac{1}{2}m\widehat{AB} = \frac{1}{2} \cdot 124^\circ = 62^\circ$

b) $m\widehat{AEB} = 2 \cdot m\angle DAB = 2 \cdot 133^\circ = 266^\circ$

Example 2: Find a, b, and c.



Solution: To find a, it is in line with 50° and 45°. The three angles add up to 180°. $50^\circ + 45^\circ + m\angle a = 180^\circ, m\angle a = 85^\circ$.

b is an inscribed angle, so its measure is half of $m\widehat{AC}$. From Theorem 9-11, $m\widehat{AC} = 2 \cdot m\angle EAC = 2 \cdot 45^\circ = 90^\circ$.

$m\angle b = \frac{1}{2} \cdot m\widehat{AC} = \frac{1}{2} \cdot 90^\circ = 45^\circ$.

To find c, you can either use the Triangle Sum Theorem or Theorem 9-11. We will use the Triangle Sum Theorem. $85^\circ + 45^\circ + m\angle c = 180^\circ, m\angle c = 50^\circ$.

From this example, we see that Theorem 9-8, from the previous section, is also true for angles formed by a tangent and chord with the vertex on the circle. If two angles, with their vertices *on* the circle, intercept the same arc then the angles are congruent.

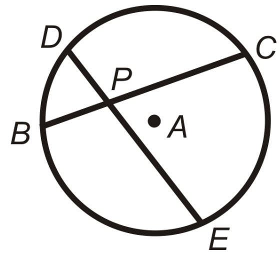
Angles

An angle is considered *inside* a circle when the vertex is somewhere inside the circle, but not on the center. All angles inside a circle are formed by two intersecting chords.

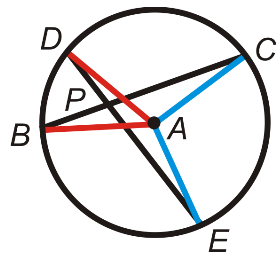
Investigation 9-7: Find the Measure of an Angle *inside* a Circle

Tools Needed: pencil, paper, compass, ruler, protractor, colored pencils (optional)

a. Draw $\odot A$ with chord $\overline{BC}$ and $\overline{DE}$. Label the point of intersection P.



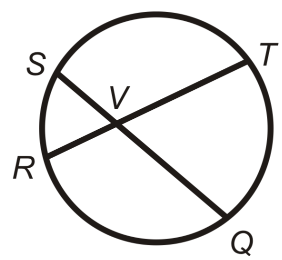
b. Draw central angles $\angle DAB$ and $\angle CAE$. Use colored pencils, if desired.



c. Using your protractor, find $m\angle DPB, m\angle DAB$, and $m\angle CAE$. What is $m\widehat{DB}$ and $m\widehat{CE}$?
d. Find $\frac{m\widehat{DB}+m\widehat{CE}}{2}$.
e. What do you notice?

Theorem 9-12: The measure of the angle formed by two chords that intersect *inside* a circle is the average of the measure of the intercepted arcs.

<u>In the picture to the left:</u>

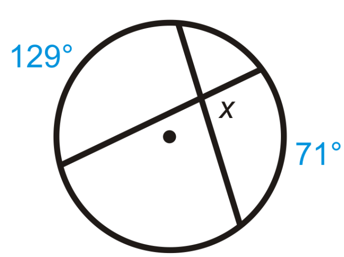


$$m\angle SVR = \frac{1}{2}\left(m\widehat{SR}+m\widehat{TQ}\right) = \frac{m\widehat{SR}+m\widehat{TQ}}{2} = m\angle TVQ$$
$$m\angle SVT = \frac{1}{2}\left(m\widehat{ST}+m\widehat{RQ}\right) = \frac{m\widehat{ST}+m\widehat{RQ}}{2} = m\angle RVQ$$

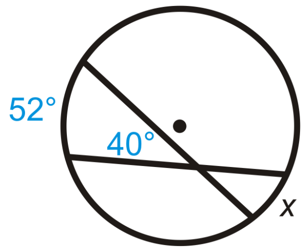
The proof of this theorem is in the review exercises.

Example 3: Find x.

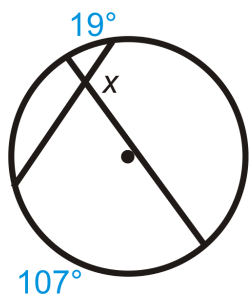
a)



b)



c)



Solution: Use Theorem 9-12 and write an equation.

a) The intercepted arcs for x are 129° and 71°.

$$x = \frac{129^\circ + 71^\circ}{2} = \frac{200^\circ}{2} = 100^\circ$$

b) Here, x is one of the intercepted arcs for 40°.

$$\begin{aligned} 40^\circ &= \frac{52^\circ + x}{2} \\ 80^\circ &= 52^\circ + x \\ 38^\circ &= x \end{aligned}$$

c) x is supplementary to the angle that the average of the given intercepted arcs. We will call this supplementary angle y.

$y = \frac{19^\circ + 107^\circ}{2} = \frac{126^\circ}{2} = 63^\circ$ This means that $x = 117^\circ; 180^\circ - 63^\circ$

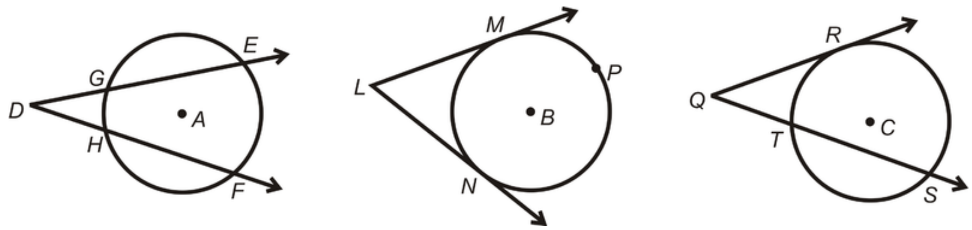
Angles

An angle is considered to be outside a circle if the vertex of the angle is outside the circle and the sides are tangents or secants. There are three types of angles that are outside a circle: an angle formed by two tangents, an angle formed by a tangent and a secant, and an angle formed by two secants. Just like an angle inside or on a circle, an angle outside a circle has a specific formula, involving the intercepted arcs.

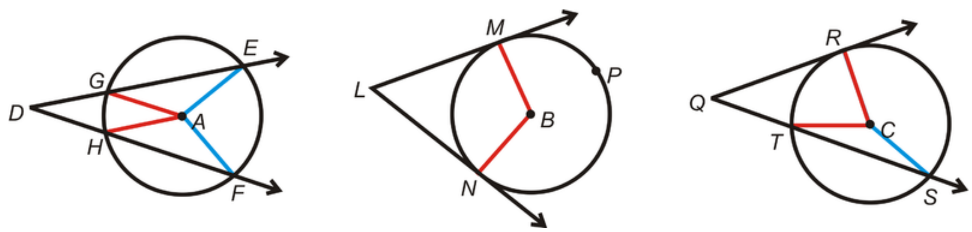
Investigation 9-8: Find the Measure of an Angle outside a Circle

Tools Needed: pencil, paper, ruler, compass, protractor, colored pencils (optional)

a. Draw three circles and label the centers A, B, and C. In $\odot A$ draw two secant rays with the same endpoint, $\overrightarrow{DE}$ and $\overrightarrow{DF}$. In $\odot B$, draw two tangent rays with the same endpoint, $\overrightarrow{LM}$ and $\overrightarrow{LN}$. In $\odot C$, draw a tangent ray and a secant ray with the same endpoint, $\overrightarrow{QR}$ and $\overrightarrow{QS}$. Label the points of intersection with the circles like they are in the pictures below.



b. Draw in all the central angles: $\angle GAH, \angle EAF, \angle MBN, \angle RCT, \angle RCS$. Then, find the measures of each of these angles using your protractor. Use color to differentiate.



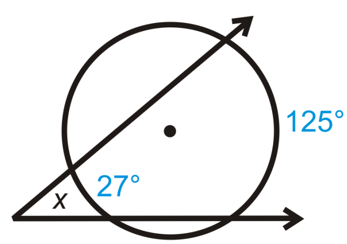
c. Find $m\angle EDF, m\angle MLN$, and $m\angle RQS$.

d. Find $\frac{m\widehat{EF}-m\widehat{GH}}{2}, \frac{m\widehat{MPN}-m\widehat{MN}}{2}$, and $\frac{m\widehat{RS}-m\widehat{RT}}{2}$. What do you notice?

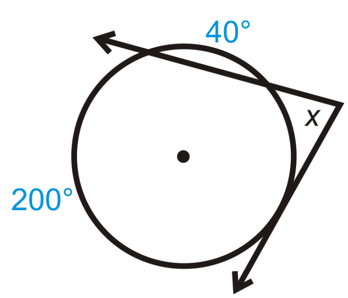
Theorem 9-13: The measure of an angle formed by two secants, two tangents, or a secant and a tangent drawn from a point outside the circle is equal to half the difference of the measures of the intercepted arcs.

Example 4: Find the measure of x.

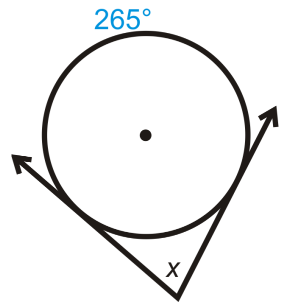
a)



b)



c)



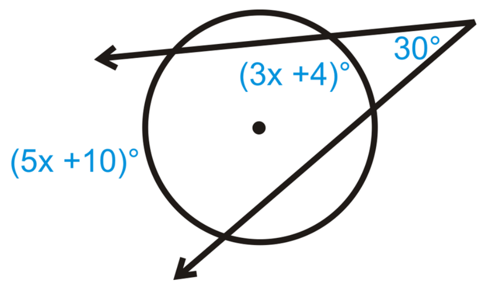
Solution: For all of the above problems we can use Theorem 9-13.

a) $x = \frac{125^\circ - 27^\circ}{2} = \frac{98^\circ}{2} = 49^\circ$

b) 40° is not the intercepted arc. Be careful! The intercepted arc is 120°, $(360^\circ - 200^\circ - 40^\circ)$. Therefore, $x = \frac{200^\circ - 120^\circ}{2} = \frac{80^\circ}{2} = 40^\circ$.

c) First, we need to find the other intercepted arc, $360^\circ - 265^\circ = 95^\circ$. $x = \frac{265^\circ - 95^\circ}{2} = \frac{170^\circ}{2} = 85^\circ$

Example 5: ***Algebra Connection*** Find the value of x. You may assume lines that look tangent, are.



Solution: Set up an equation using Theorem 9-13.

$$\begin{aligned}
\frac{(5x+10)^\circ - (3x+4)^\circ}{2} &= 30^\circ \\
(5x+10)^\circ - (3x+4)^\circ &= 60^\circ \\
5x + 10^\circ - 3x - 4^\circ &= 60^\circ \\
2x + 6^\circ &= 60^\circ \\
2x &= 54^\circ \\
x &= 27^\circ
\end{aligned}$$

Know What? Revisited If 178° of the Earth is exposed to the sun, then the angle at which the sun's rays hit the Earth is 2°. From Theorem 9-13, these two angles are supplementary. From this, we also know that the other 182° of the Earth is not exposed to sunlight and it is probably night time.

Review Questions

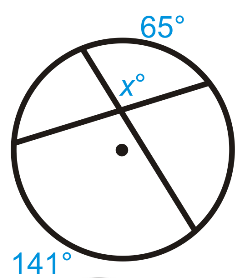
1. Draw two secants that intersect:
    a. inside a circle.
    b. on a circle.
    c. outside a circle.

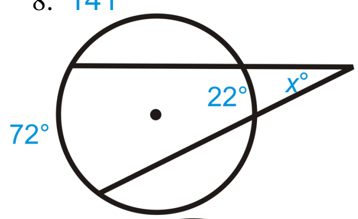
2. Can two tangent lines intersect inside a circle? Why or why not?
3. Draw a tangent and a secant that intersect:
   a. on a circle.
   b. outside a circle.

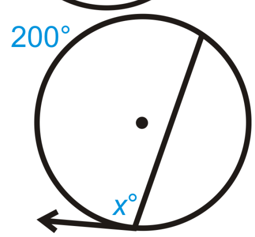
Fill in the blanks.

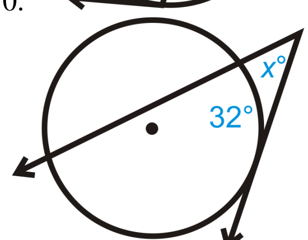
4. If the vertex of an angle is on the _______________ of a circle, then its measure is _______________ to the intercepted arc.
5. If the vertex of an angle is _______________ a circle, then its measure is the average of the _____________-_____ arcs.
6. If the vertex of an angle is ________ a circle, then its measure is ______________ the intercepted arc.
7. If the vertex of an angle is ____________ a circle, then its measure is ___________ the difference of the intercepted arcs.

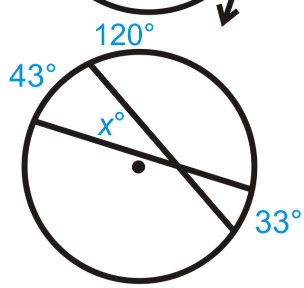
For questions 8-19, find the value of the missing variable(s).

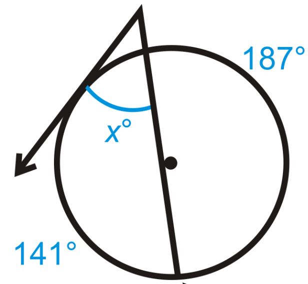
8. 

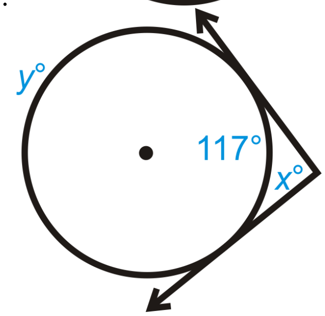
9. 

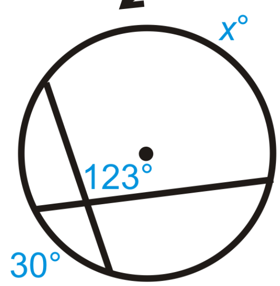
10. 

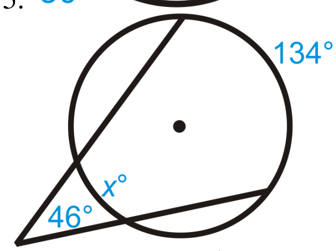
11. 

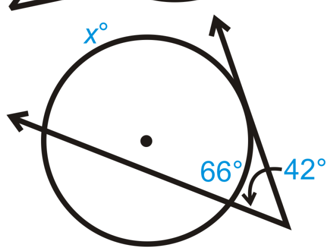
12. 

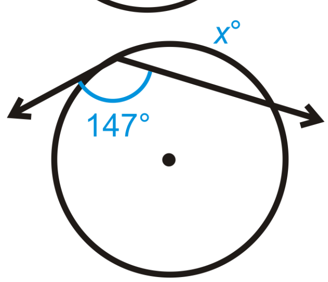
13. 

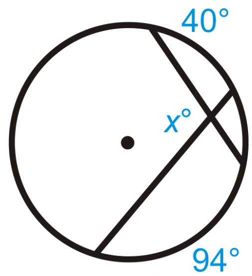
14. 

15. 

16. 

17. 

18. 



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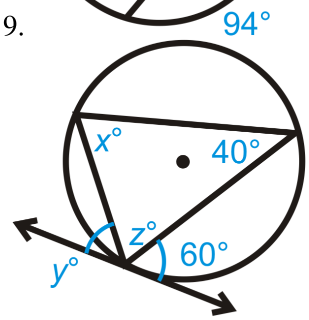

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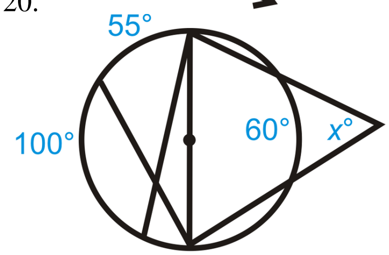

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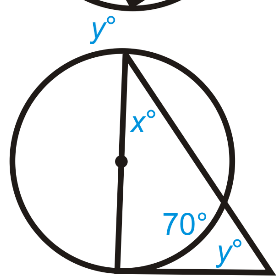
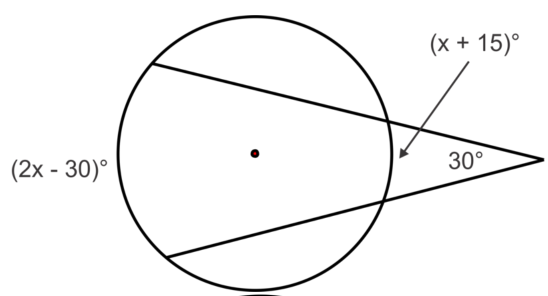


22.

Algebra Connection Solve for the variable(s).



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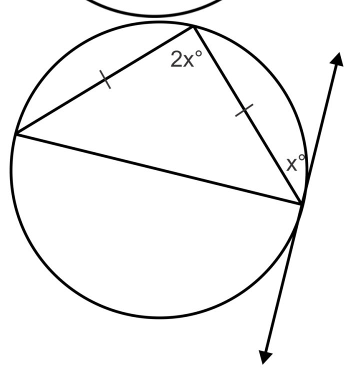

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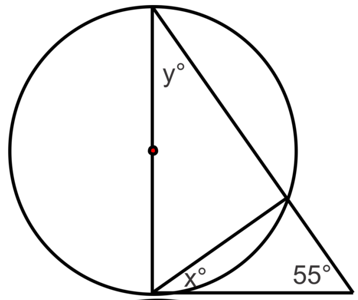
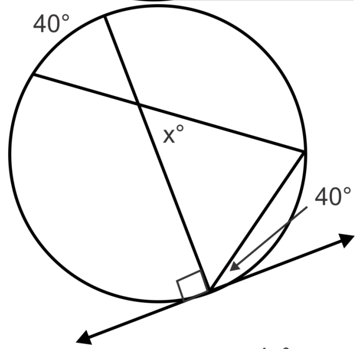
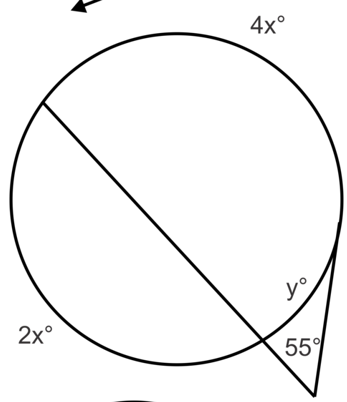
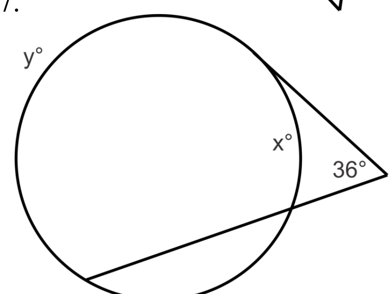
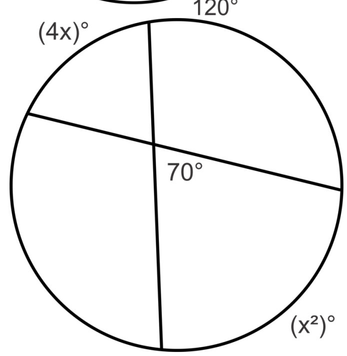
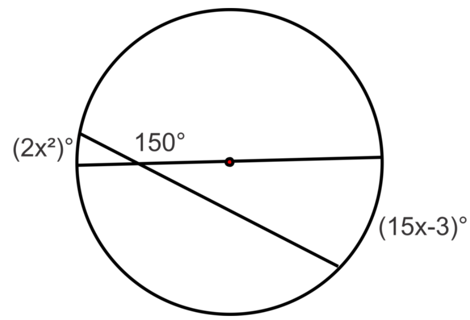
25. 

26. 

27. 

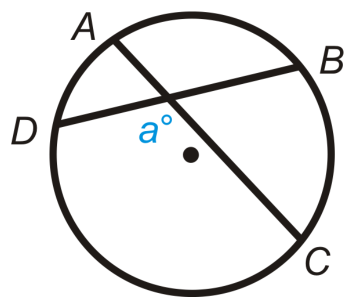
28. 

29. 



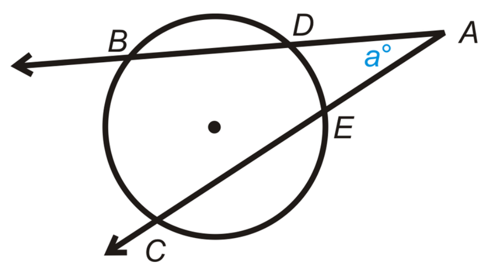
30.

31. Prove Theorem 9-12.



Given: Intersecting chords $\overline{AC}$ and $\overline{BD}$. Prove: $m\angle a = \frac{1}{2}\left(m\widehat{DC} + m\widehat{AB}\right)$ *HINT:* Draw $\overline{BC}$ and use inscribed angles.

32. Prove Theorem 9-13.



Given: Secant rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$Prove: $m\angle a = \frac{1}{2}\left(m\widehat{BC} - m\widehat{DE}\right)$ *HINT:* Draw $\overline{BE}$ and use inscribed angles.

Review Queue Answers

a. $m\angle OML = m\angle OPL = 90^\circ$ because a tangent line and a radius drawn to the point of tangency are perpendicular.

b. $$165^\circ + m\angle OML + m\angle OPL + m\angle MLP = 360^\circ$$
$$165^\circ + 90^\circ + 90^\circ + m\angle MLP = 360^\circ$$
$$m\angle MLP = 15^\circ$$

c. $m\widehat{MNP} = 360^\circ - 165^\circ = 195^\circ$

d. $\frac{195^\circ - 165^\circ}{2} = \frac{30^\circ}{2} = 15^\circ$, this is the same as $m\angle MLP$.

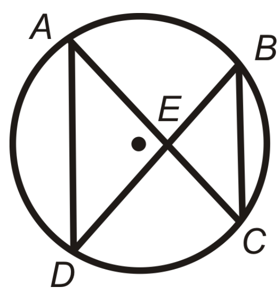
12.6 Segments of Chords, Secants, and Tangents

Learning Objectives

- Find the lengths of segments associated with circles.

Review Queue

a. What can you say about $m\angle DAC$ and $m\angle DBC$? What theorem do you use?

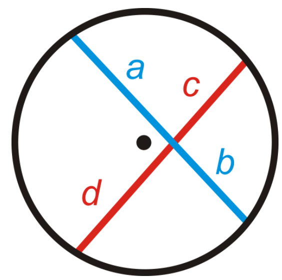


b. What do you know about $m\angle AED$ and $m\angle BEC$? Why?
c. Is $\triangle AED \sim \triangle BEC$? How do you know?
d. If $AE = 8, ED = 7$, and $BE = 6$, find EC.
e. If $\overline{AD}$ and $\overline{BC}$ are not in the circle, would the ratios from #4 still be valid?

Know What? As you know, the moon orbits the earth. At a particular time, the moon is 238,857 miles from Beijing, China. On the same line, Yukon is 12,451 miles from Beijing. Drawing another line from the moon to Cape Horn (the southernmost point of South America), we see that Jakarta, Indonesia is collinear. If the distance from Cape Horn to Jakarta is 9849 miles, what is the distance from the moon to Jakarta?

Segments from Chords

In the Review Queue above, we have two chords that intersect inside a circle. The two triangles are similar, making the sides of each triangle in proportion with each other. If we remove $\overline{AD}$ and $\overline{BC}$ the ratios between $\overline{AE}, \overline{EC}, \overline{DE}$, and $\overline{EB}$ will still be the same. This leads us to our first theorem.

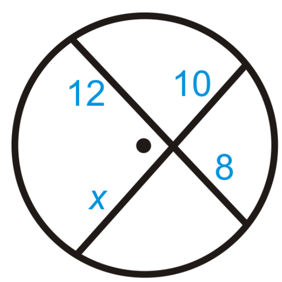


Theorem 9-14: If two chords intersect inside a circle so that one is divided into segments of length a and b and the other into segments of length c and d then $ab = cd$.

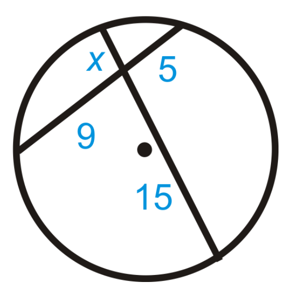
The product of the segments of one chord is equal to the product of segments of the second chord.

Example 1: Find x in each diagram below.

a)



b)



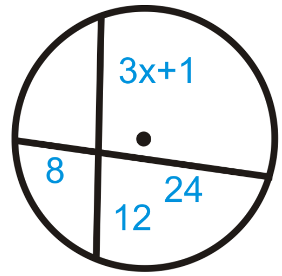
Solution: Use the ratio from Theorem 9-13. The product of the segments of one chord is equal to the product of the segments of the other.

a) $12 \cdot 8 = 10 \cdot x$
$96 = 10x$
$9.6 = x$

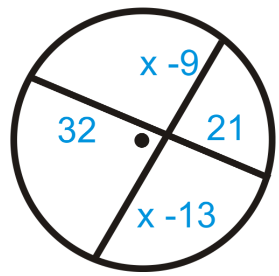
b) $x \cdot 15 = 5 \cdot 9$
$15x = 45$
$x = 3$

Example 2: ***Algebra Connection*** Solve for x.

a)



b)



Solution: Again, we can use Theorem 9-13. Set up an equation and solve for x.

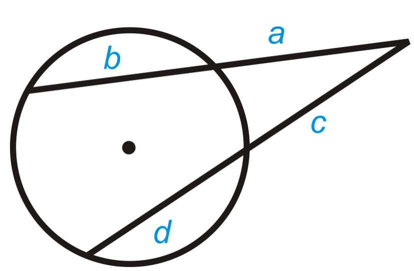
a) $8 \cdot 24 = (3x+1) \cdot 12$
$192 = 36x + 12$
$180 = 36x$
$5 = x$

b) $32 \cdot 21 = (x-9)(x-13)$
$672 = x^2 - 22x + 117$
$0 = x^2 - 22x - 555$
$0 = (x-37)(x+15)$
$x = 37, -15$

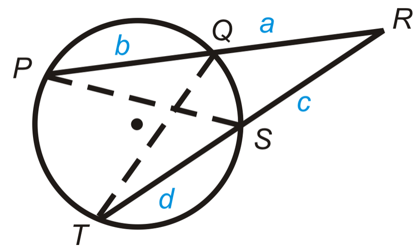
However, $x \neq -15$ because length cannot be negative, so $x = 37$.

Segments from Secants

In addition to forming an angle outside of a circle, the circle can divide the secants into segments that are proportional with each other.



If we draw in the intersecting chords, we will have two similar triangles.



From the inscribed angles and the Reflexive Property $(\angle R \cong \angle R), \triangle PRS \sim \triangle TRQ$.

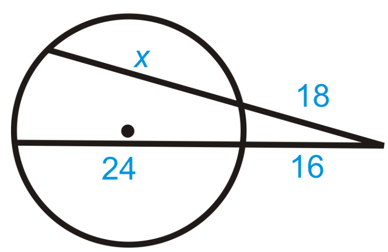
Because the two triangles are similar, we can set up a proportion between the corresponding sides. Then, cross-multiply. $\frac{a}{c+d} = \frac{c}{a+b} \Rightarrow a(a+b) = c(c+d)$

Theorem 9-15: If two secants are drawn from a common point outside a circle and the segments are labeled as above, then $a(a+b) = c(c+d)$.

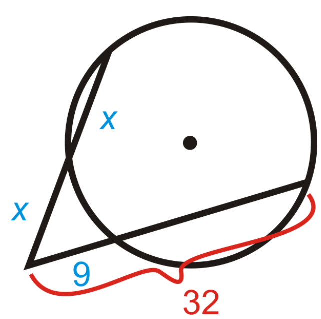
In other words, the product of the outer segment and the whole of one secant is equal to the product of the outer segment and the whole of the other secant.

Example 3: Find the value of the missing variable.

a)



b)



Solution: Use Theorem 9-15 to set up an equation. For both secants, you multiply the outer portion of the secant by the whole.

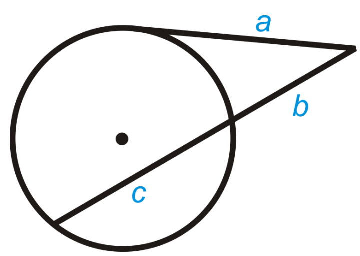
a) $18 \cdot (18+x) = 16 \cdot (16+24)$
$324 + 18x = 256 + 384$
$18x = 316$
$x = 17\frac{5}{9}$

b) $x \cdot (x+x) = 9 \cdot 32$
$2x^2 = 288$
$x^2 = 144$
$x = 12$

$x \neq -12$ because length cannot be negative.

Segments from Secants and Tangents

If a tangent and secant meet at a common point outside a circle, the segments created have a similar relationship to that of two secant rays in Example 3. Recall that the product of the outer portion of a secant and the whole is equal to the same of the other secant. If one of these segments is a tangent, it will still be the product of the outer portion and the whole. However, for a tangent line, the outer portion and the whole are equal.

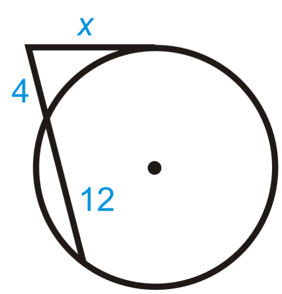


Theorem 9-16: If a tangent and a secant are drawn from a common point outside the circle (and the segments are labeled like the picture to the left), then $a^2 = b(b+c)$.

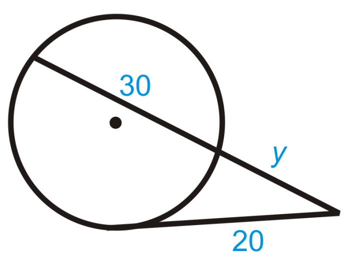
This means that the product of the outside segment of the secant and the whole is equal to the square of the tangent segment.

Example 4: Find the value of the missing segment.

a)



b)



Solution: Use Theorem 9-16. Square the tangent and set it equal to the outer part times the whole secant.

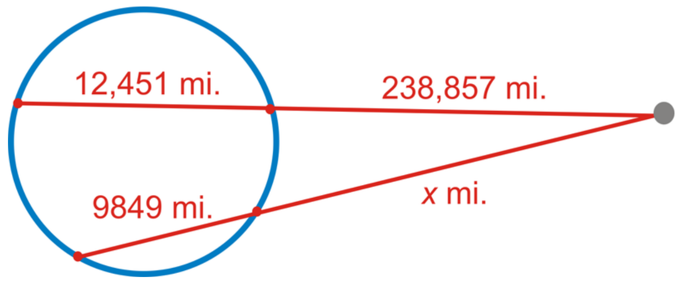
a) $x^2 = 4(4+12)$
$x^2 = 4 \cdot 16 = 64$
$x = 8$

b) $20^2 = y(y+30)$
$400 = y^2 + 30y$
$0 = y^2 + 30y - 400$

$0 = (y+40)(y-10)$
$y = \cancel{-40}, 10$

When you have to factor a quadratic equation to find an answer, always eliminate the negative answer(s). Length is never negative.

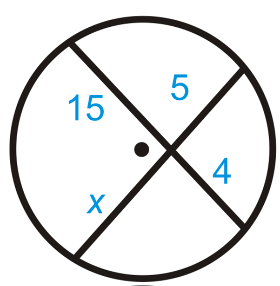
Know What? Revisited The given information is to the left. Let's set up an equation using Theorem 9-15.

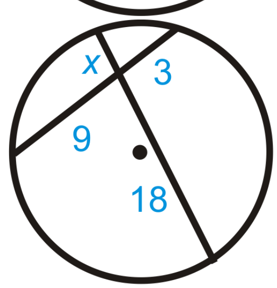


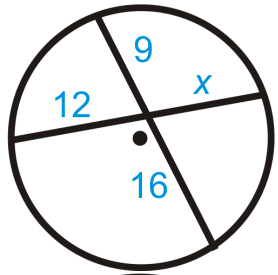
$$\begin{aligned} 238857 \cdot 251308 &= x \cdot (x+9849) \\ 60026674956 &= x^2 + 9849x \\ 0 &= x^2 + 9849x - 60026674956 \\ \text{Use the Quadratic Formula } x &\approx \frac{-9849 \pm \sqrt{9849^2 - 4(-60026674956)}}{2} \\ x &\approx 240128.4 \text{ miles} \end{aligned}$$

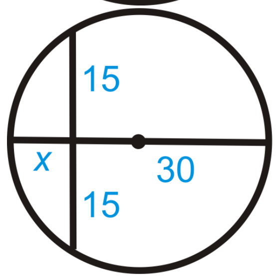
Review Questions

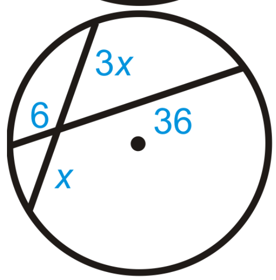
Find x in each diagram below. Simplify any radicals.

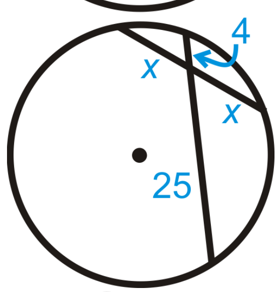
1. 

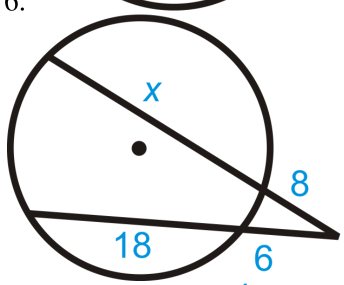
2. 

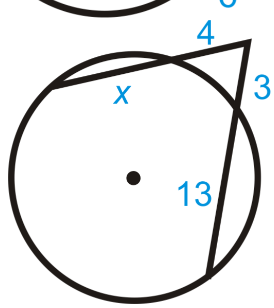
3. 

4. 

5. 

6. 

7. 

8. 

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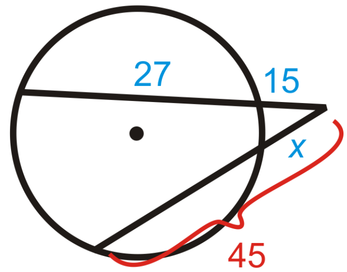

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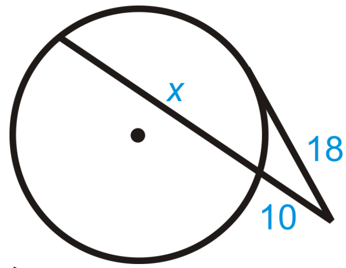

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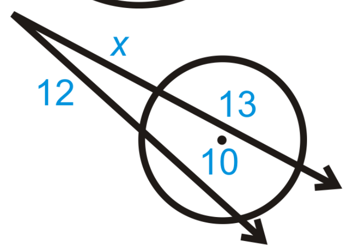

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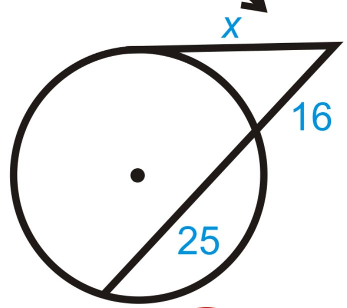

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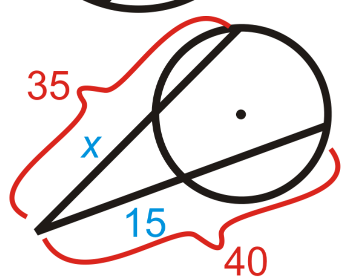

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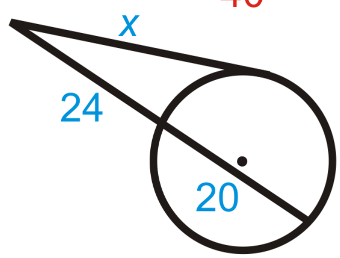

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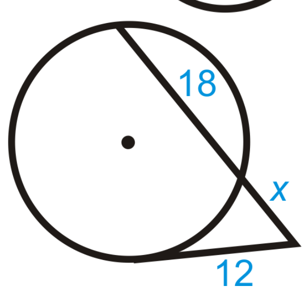

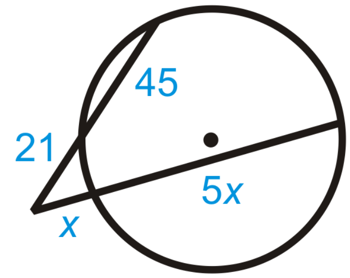

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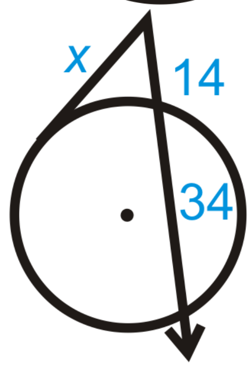

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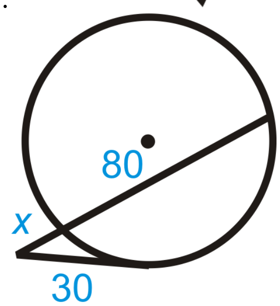
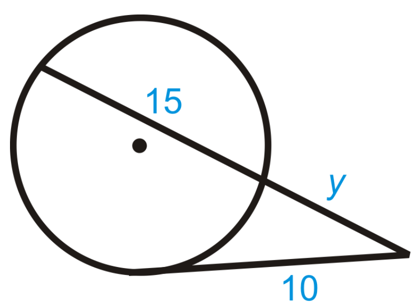


18.

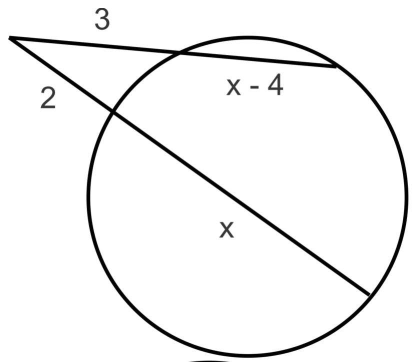
19. ***Error Analysis*** Describe and correct the error in finding y.



$$10 \cdot 10 = y \cdot 15y$$
$$100 = 15y^2$$
$$\frac{20}{3} = y^2$$
$$\frac{2\sqrt{15}}{3} = y \quad \leftarrow y \text{ is not correct}$$

20. Suzie found a piece of a broken plate. She places a ruler across two points on the rim, and the length of the chord is found to be 6 inches. The distance from the midpoint of this chord to the nearest point on the rim is found to be 1 inch. Find the diameter of the plate.

Algebra Connection For problems 21-30, solve for x.

21.


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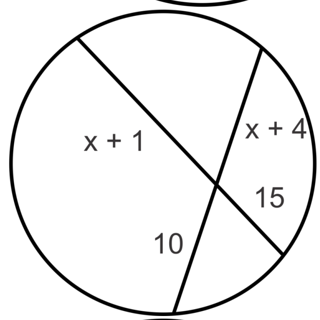

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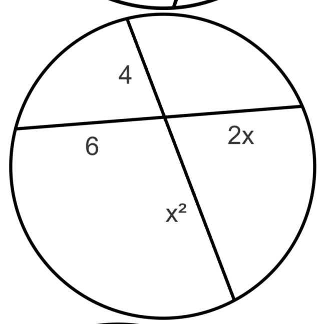

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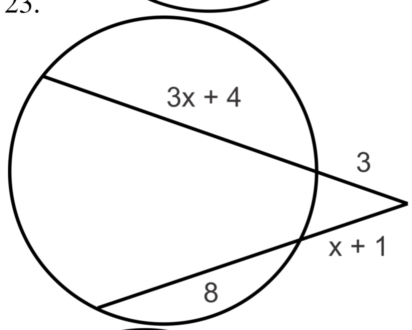

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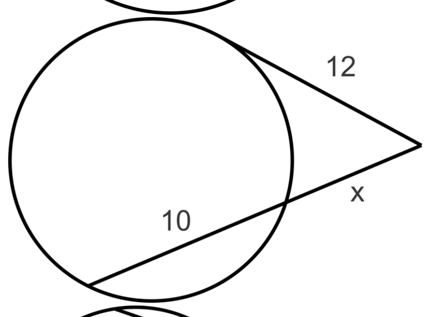

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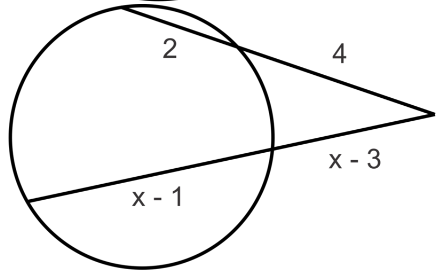

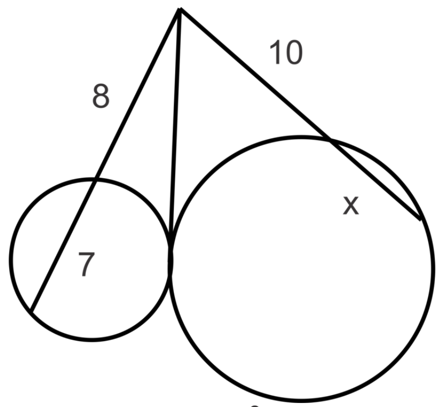

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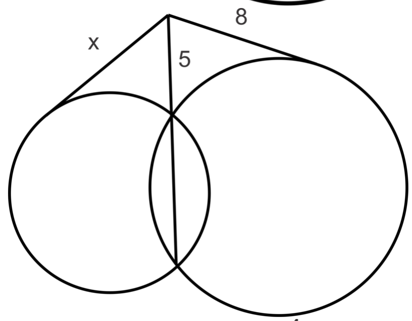

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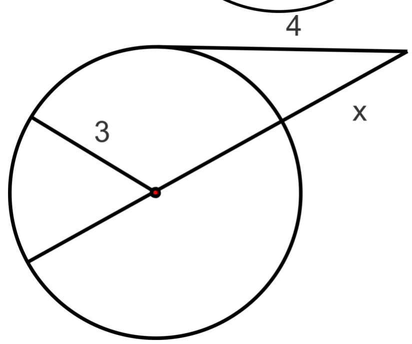
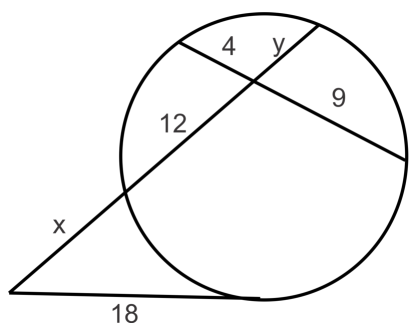


29.

30. Find x and y.



Review Queue Answers

a. $m\angle DAC = m\angle DBC$ by Theorem 9-8, they are inscribed angles and intercept the same arc.
b. $m\angle AED = m\angle BEC$ by the Vertical Angles Theorem.
c. Yes, by AA Similarity Postulate.
d. $\frac{8}{6} = \frac{7}{EC}$
$8 \cdot EC = 42$
$EC = \frac{21}{4} = 5.25$
e. Yes, the EC would be the same and the ratio would still be valid.

12.7 Circles in the Coordinate Plane

Here you'll learn how to find the standard equation for circles given their radius and center. You'll also graph circles in the coordinate plane.

What if you were given the length of the radius of a circle and the coordinates of its center? How could you write the equation of the circle in the coordinate plane? After completing this Concept, you'll be able to write the standard equation of a circle.

Watch This

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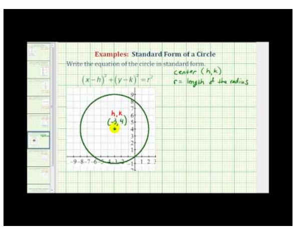
Click image to the left for more content.

Graphing Circles CK-12



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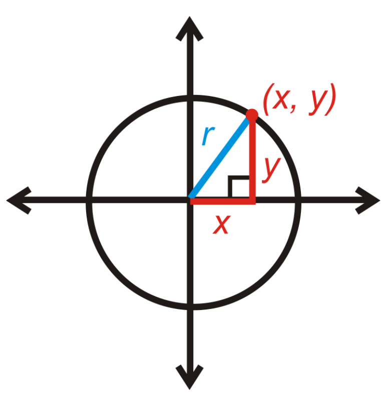
James Sousa: Write the Standard Form of a Circle

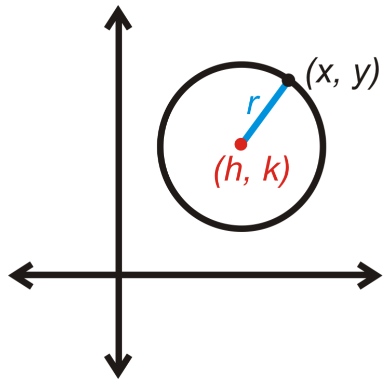
Guidance

Recall that a circle is the set of all points in a plane that are the same distance from the center. This definition can be used to find an equation of a circle in the coordinate plane.



Let's start with the circle centered at (0, 0). If (x,y) is a point on the circle, then the distance from the center to this point would be the radius, r. x is the horizontal distance and y is the vertical distance. This forms a right triangle. From the Pythagorean Theorem, the equation of a circle *centered at the origin* is $x^2+y^2=r^2$.

The center does not always have to be on (0, 0). If it is not, then we label the center (h,k). We would then use the Distance Formula to find the length of the radius.



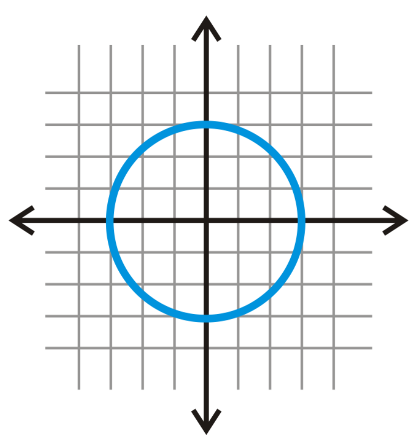
$$r=\sqrt{(x-h)^2+(y-k)^2}$$

If you square both sides of this equation, then you would have the standard equation of a circle. **The standard equation of a circle with center (h,k) and radius r is $r^2=(x-h)^2+(y-k)^2$.**

Example A

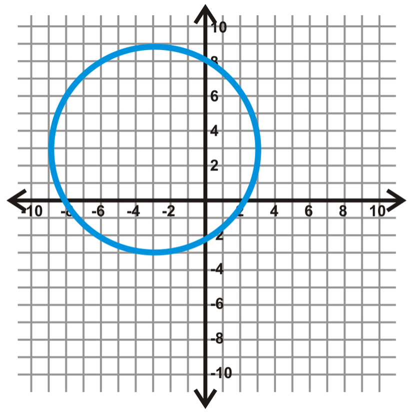
Graph $x^2+y^2=9$.

The center is (0, 0). Its radius is the square root of 9, or 3. Plot the center, plot the points that are 3 units to the right, left, up, and down from the center and then connect these four points to form a circle.

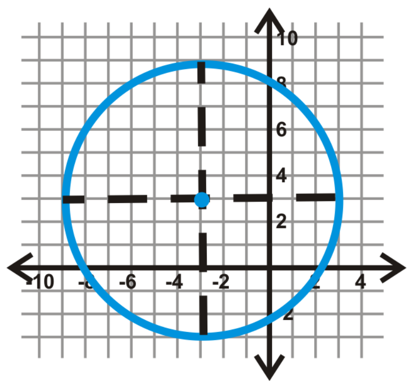


Example B

Find the equation of the circle below.



First locate the center. Draw in the horizontal and vertical diameters to see where they intersect.



From this, we see that the center is (-3, 3). If we count the units from the center to the circle on either of these diameters, we find $r = 6$. Plugging this into the equation of a circle, we get: $(x-(-3))^2 + (y-3)^2 = 6^2$ or $(x+3)^2 + (y-3)^2 = 36$.

Example C

Determine if the following points are on $(x+1)^2 + (y-5)^2 = 50$.

a) (8, -3)

b) (-2, -2)

Plug in the points for x and y in $(x+1)^2 + (y-5)^2 = 50$.

a)

$$\begin{aligned}(8+1)^2 + (-3-5)^2 &= 50 \\ 9^2 + (-8)^2 &= 50 \\ 81 + 64 &\neq 50\end{aligned}$$

(8, -3) is *not* on the circle

b)

$$\begin{aligned}(-2+1)^2 + (-2-5)^2 &= 50 \\ (-1)^2 + (-7)^2 &= 50 \\ 1 + 49 &= 50\end{aligned}$$

(-2, -2) is on the circle

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Graphing Circles CK-12

Guided Practice

Find the center and radius of the following circles.

1. $(x-3)^2+(y-1)^2=25$
2. $(x+2)^2+(y-5)^2=49$

3. Find the equation of the circle with center (4, -1) and which passes through (-1, 2).

Answers:

1. Rewrite the equation as $(x-3)^2+(y-1)^2=5^2$. The center is (3, 1) and $r=5$.

2. Rewrite the equation as $(x-(-2))^2+(y-5)^2=7^2$. The center is (-2, 5) and $r=7$.

Keep in mind that, due to the minus signs in the formula, the coordinates of the center have the ***opposite signs*** of what they may initially appear to be.

3. First plug in the center to the standard equation.

$$
\begin{aligned}
(x-4)^2+(y-(-1))^2&=r^2\\
(x-4)^2+(y+1)^2&=r^2
\end{aligned}
$$

Now, plug in (-1, 2) for x and y and solve for r.

$$
\begin{aligned}
(-1-4)^2+(2+1)^2&=r^2\\
(-5)^2+(3)^2&=r^2\\
25+9&=r^2\\
34&=r^2
\end{aligned}
$$

Substituting in 34 for r^2, the equation is $(x-4)^2+(y+1)^2=34$.

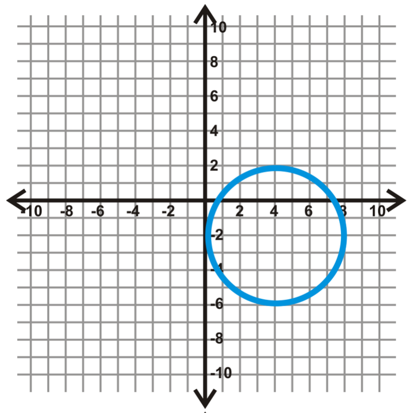
Practice

Find the center and radius of each circle. Then, graph each circle.

1. $(x+5)^2+(y-3)^2=16$
2. $x^2+(y+8)^2=4$
3. $(x-7)^2+(y-10)^2=20$

4. $(x+2)^2+y^2=8$

Find the equation of the circles below.



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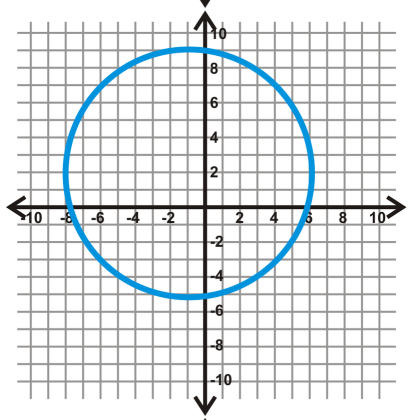

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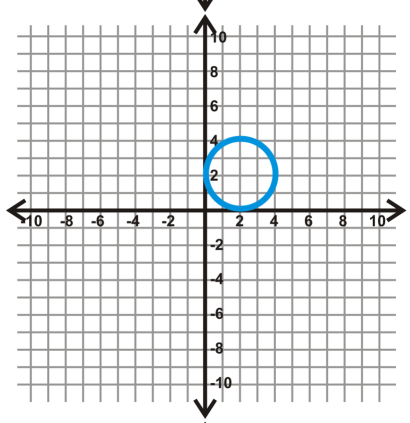

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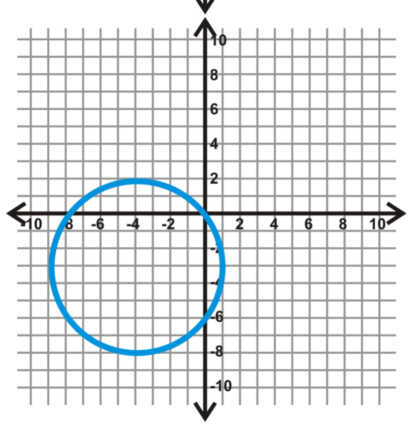


8.

9. Is (-7, 3) on $(x+1)^2+(y-6)^2=45$?
10. Is (9, -1) on $(x-2)^2+(y-2)^2=60$?

11. Is (-4, -3) on $(x+3)^2+(y-3)^2=37$?
12. Is (5, -3) on $(x+1)^2+(y-6)^2=45$?

Find the equation of the circle with the given center and point on the circle.

13. center: (2, 3), point: (-4, -1)
14. center: (10, 0), point: (5, 2)
15. center: (-3, 8), point: (7, -2)
16. center: (6, -6), point: (-9, 4)

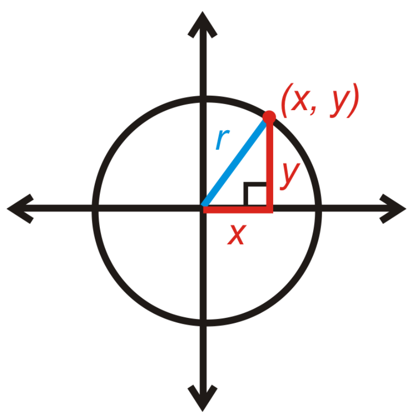
12.8 Extension: Writing and Graphing the Equations of Circles

Learning Objectives

- Graph a circle.
- Find the equation of a circle in the coordinate plane.
- Find the radius and center, given the equation of a circle and vice versa.
- Find the equation of a circle, given the center and a point on the circle.

Graphing a Circle in the Coordinate Plane

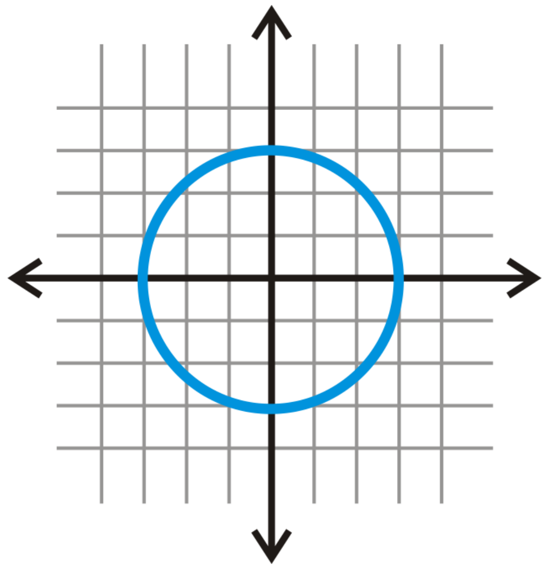
Recall that the definition of a circle is the set of all points that are the same distance from a point, called the center. This definition can be used to find an equation of a circle in the coordinate plane.



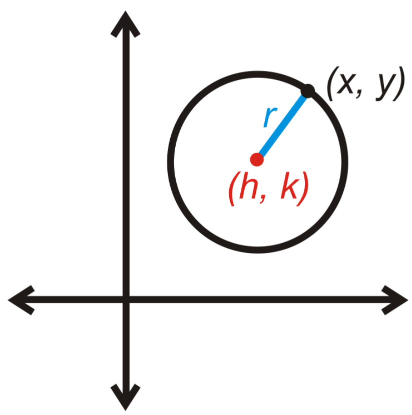
Let's start with the circle centered at the origin, (0, 0). If (x,y) is a point on the circle, then the distance from the center to this point would be the radius, r. x is the horizontal distance of the coordinate and y is the vertical distance. Drawing those in, we form a right triangle. Therefore, the equation of a circle, *centered at the origin* is $x^2+y^2=r^2$, by the Pythagorean Theorem.

Example 1: Graph $x^2+y^2=9$.

Solution: This circle is centered at the origin. It's radius is the square root of 9, or 3. The easiest way to graph a circle is to plot the center, and then go out 3 units in every direction and connect them to form a circle.



The center does not always have to be on (0, 0). If it is not, then we label the center (h,k) and would use the distance formula to find the length of the radius.



$$r = \sqrt{(x-h)^2 + (y-k)^2}$$

If you square both sides of this equation, then we would have the standard equation of a circle.

Standard Equation of a Circle: The standard equation of a circle with center (h,k) and radius r is $r^2 = (x-h)^2 + (y-k)^2$.

Example 2: Find the center and radius of the following circles.

a) $(x-3)^2 + (y-1)^2 = 25$

b) $(x+2)^2 + (y-5)^2 = 49$

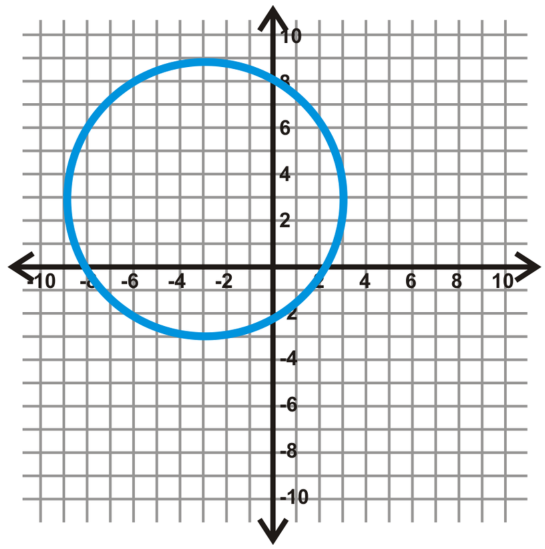
Solution:

a) Rewrite the equation as $(x-3)^2 + (y-1)^2 = 5^2$. Therefore, the center is (3, 1) and the radius is 5.

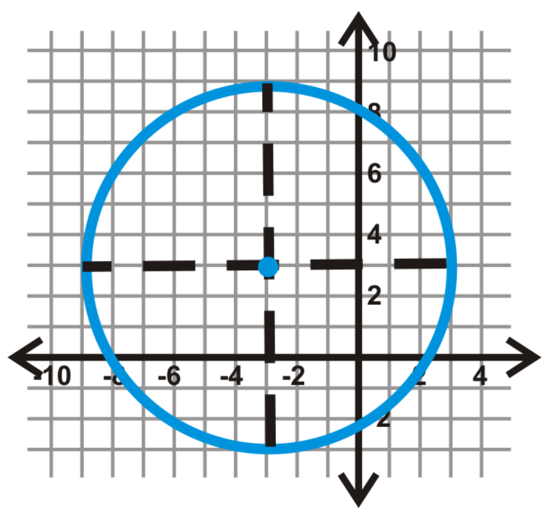
b) Rewrite the equation as $(x-(-2))^2 + (y-5)^2 = 7^2$. From this, the center is (-2, 5) and the radius is 7.

When finding the center of a circle always take the ***opposite sign*** of what the value is in the equation.

Example 3: Find the equation of the circle below.



Solution: First locate the center. Draw in a couple diameters. It is easiest to use the horizontal and vertical diameters.



From the intersecting diameters, we see that the center is (-3, 3). If we count the units from the center to the circle on either of these diameters, we find that the radius is 6. Plugging this information into the equation of a circle, we get $(x-(-3))^2+(y-3)^2=6^2$ or $(x+3)^2+(y-3)^2=36$.

Finding the Equation of a Circle

Example 4: Find the equation of the circle with center (4, -1) and passes through (-1, 2).

Solution: To find the equation, first plug in the center to the standard equation.

$(x-4)^2+(y-(-1))^2=r^2$ or $(x-4)^2+(y+1)^2=r^2$

Now, plug in (-1, 2) for x and y and solve for r.

$$\begin{aligned}(-1-4)^2+(2+1)^2 &= r^2\\(-5)^2+(3)^2 &= r^2\\25+9 &= r^2\\34 &= r^2\end{aligned}$$

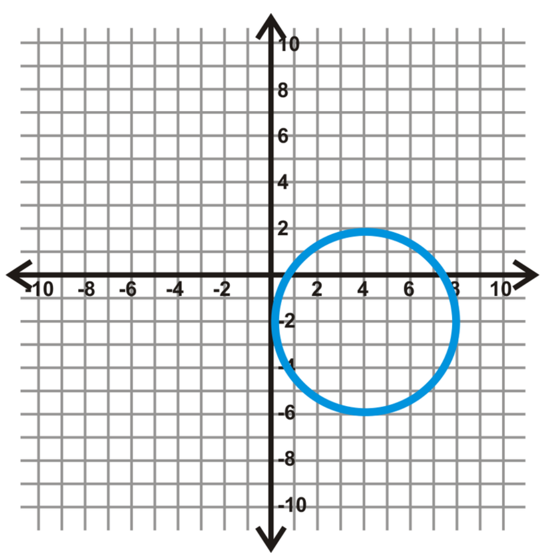
At this point, we don't need to solve for r because r^2 is what is in the equation. Substituting in 34 for r^2, we have $(x-4)^2+(y+1)^2=34$.

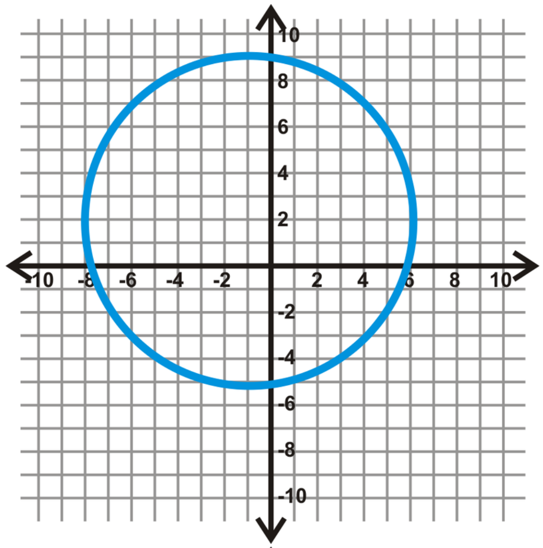
Review Questions

Find the center and radius of each circle. Then, graph each circle.

1. $(x+5)^2+(y-3)^2=16$
2. $x^2+(y+8)^2=4$
3. $(x-7)^2+(y-10)^2=20$
4. $(x+2)^2+y^2=8$

Find the equation of the circles below.

5. 



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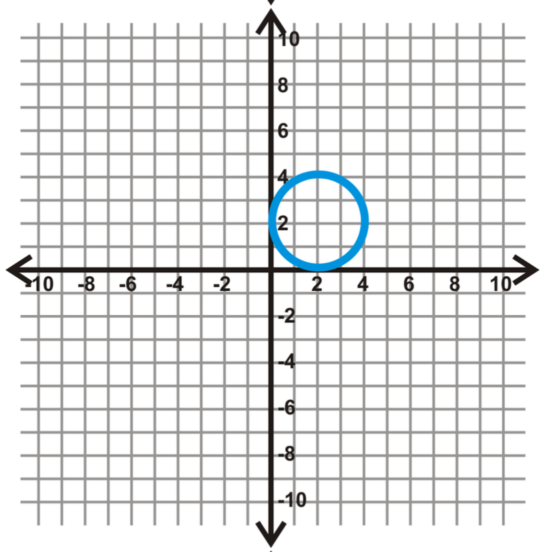

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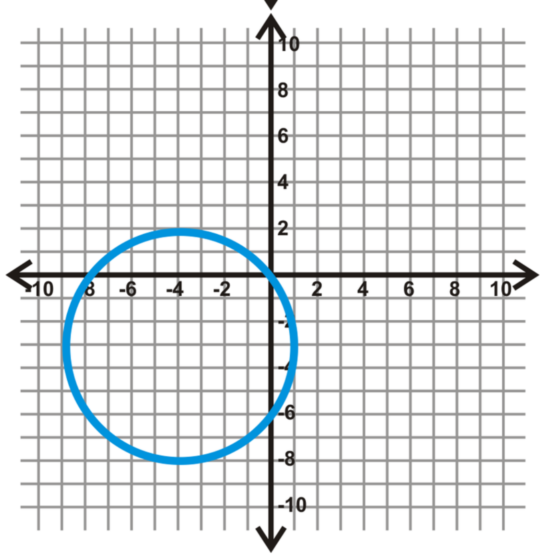

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9. Determine if the following points are on $(x+1)^2 + (y-6)^2 = 45$.

    a. (2, 0)
    b. (-3, 4)
    c. (-7, 3)

Find the equation of the circle with the given center and point on the circle.

10. center: (2, 3), point: (-4, -1)
11. center: (10, 0), point: (5, 2)
12. center: (-3, 8), point: (7, -2)
13. center: (6, -6), point: (-9, 4)
14. Now let's find the equation of a circle using three points on the circle. Do you remember how we found the center and radius of a circle given three points on the circle in problem 30 of Section 9-3? We used the fact that the perpendicular bisector of any chord in the circle will pass through the center. By finding the perpendicular bisectors of two different chords and their intersection we can find the center of the circle. Then we can use the distance formula with the center and a point on the circle to find the radius. Finally, we will write the equation. Given the points $A(-12,-21), B(2,27)$ and $C(19,10)$ on the circle (an arc could be drawn through these points from A to C), follow the steps below.
    a. Since the perpendicular bisector passes through the midpoint of a segment we must first find the midpoint between A and B.
    b. Now the perpendicular line must have a slope that is the opposite reciprocal of the slope of $\overleftrightarrow{AB}$. Find the slope of $\overleftrightarrow{AB}$ and then its opposite reciprocal.
    c. Finally, you can write the equation of the perpendicular bisector of $\overline{AB}$ using the point you found in part a and the slope you found in part b.
    d. Repeat steps a-c for chord $\overline{BC}$.
    e. Now that we have the two perpendicular bisectors of the chord we can find their intersection. Solve the system of linear equations to find the center of the circle.
    f. Find the radius of the circle by finding the distance from the center (point found in part *e*) to any of the three given points on the circle.
    g. Now, use the center and radius to write the equation of the circle.

Find the equations of the circles which contain three points in problems 15 and 16.

15. $A(-2,5), B(5,6)$ and $C(6,-1)$
16. $A(-11,-14), B(5,16)$ and $C(12,9)$

12.9 Chapter 12 Review

Keywords Theorems

Circle
The set of all points that are the same distance away from a specific point

Center
The set of all points that are the same distance away from a specific point, called the **center.**

Radius
The distance from the center to the circle.

Chord
A line segment whose endpoints are on a circle.

Diameter
A chord that passes through the center of the circle.

Secant
A line that intersects a circle in two points.

Tangent
A line that intersects a circle in exactly one point.

Point of Tangency
The point where the tangent line touches the circle.

Congruent Circles
Two circles with the same radius, but different centers.

Concentric Circles
When two circles have the same center, but different radii.

Tangent to a Circle Theorem
A line is tangent to a circle if and only if the line is perpendicular to the radius drawn to the point of tangency.

Theorem 9-2
If two tangent segments are drawn from the same external point, then the segments are equal.

Central Angle
The angle formed by two radii of the circle with its vertex at the center of the circle.

Arc
A section of the circle.

Semicircle
An arc that measures 180°.

Minor Arc
An arc that is less than 180°.

Major Arc
An arc that is greater than 180°. ***Always*** use 3 letters to label a major arc.

Congruent Arcs
Two arcs are congruent if their central angles are congruent.

Arc Addition Postulate
The measure of the arc formed by two adjacent arcs is the sum of the measures of the two

Theorem 9-3
In the same circle or congruent circles, minor arcs are congruent if and only if their corresponding chords are congruent.

Theorem 9-4
The perpendicular bisector of a chord is also a diameter.

Theorem 9-5
If a diameter is perpendicular to a chord, then the diameter bisects the chord and its corresponding arc.

Theorem 9-6
In the same circle or congruent circles, two chords are congruent if and only if they are equidistant from the center.

Inscribed Angle
An angle with its vertex is the circle and its sides contain chords.

Intercepted Arc
The arc that is on the interior of the inscribed angle and whose endpoints are on the angle.

Inscribed Angle Theorem
The measure of an inscribed angle is half the measure of its intercepted arc.

Theorem 9-8
Inscribed angles that intercept the same arc are congruent.

Theorem 9-9
An angle that intercepts a semicircle is a right angle.

Inscribed Polygon
A polygon where every vertex is on a circle.

Theorem 9-10
A quadrilateral is inscribed in a circle if and only if the opposite angles are supplementary.

Theorem 9-11

The measure of an angle formed by a chord and a tangent that intersect on the circle is half the measure of the intercepted arc.

Theorem 9-12

The measure of the angle formed by two chords that intersect *inside* a circle is the average of the measure of the intercepted arcs.

Theorem 9-13

The measure of an angle formed by two secants, two tangents, or a secant and a tangent drawn from a point outside the circle is equal to half the difference of the measures of the intercepted arcs.

Theorem 9-14

The product of the segments of one chord is equal to the product of segments of the second chord.

Theorem 9-15

If two secants are drawn from a common point outside a circle and the segments are labeled as above, then $a(a+b) = c(c+d)$.

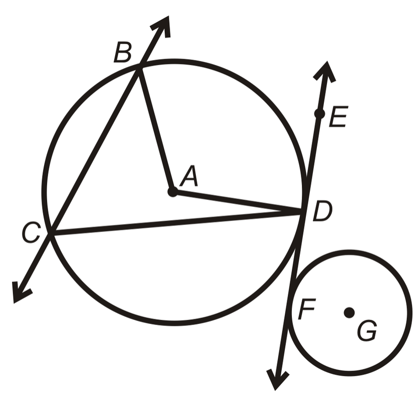
Theorem 9-16

If a tangent and a secant are drawn from a common point outside the circle (and the segments are labeled like the picture to the left), then $a^2 = b(b+c)$.

Standard Equation of a Circle

The standard equation of a circle with center (h,k) and radius r is $r^2 = (x-h)^2 + (y-k)^2$.

Vocabulary



Match the description with the correct label.

1. minor arc - A. $\overline{CD}$
2. chord - B. $\overline{AD}$
3. tangent line - C. $\overleftrightarrow{CB}$
4. central angle - D. $\overleftrightarrow{EF}$
5. secant - E. A
6. radius - F. D
7. inscribed angle - G. $\angle BAD$
8. center - H. $\angle BCD$
9. major arc - I. $\widehat{BD}$
10. point of tangency - J. $\widehat{BCD}$

Texas Instruments Resources

In the CK-12 Texas Instruments Geometry FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9694 .